ON STOCHASTIC DOMINANCE OPTION BOUNDS IN DISCRETE AND CONTINUOUS SPACE AND TIME WITH STOCHASTIC AND DETERMINISTIC VOLATILITY AND PRICING WITH CONSTANT RELATIVE RISK AVERSION

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Abstract

By

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This thesis makes original contributions to the field of asset pricing, which is a field dedicated to describing the prices of financial instruments and their characteristics. The prices of these financial instruments are determined by the behavior of investors who buy and sell them, and so asset pricing is ultimately done by modeling the behavior of investors. One method for achieving this is through the framework of stochastic dominance. This thesis specifically deals with a specific class of financial instruments called European options and reviews the literature on stochastic dominance option pricing and discusses new methods for finding stochastic dominance bounds on options in both discrete and continuous time under both deterministic and stochastic volatility. The results presented here extends the works of Ritchken and Kuo[29] and Perrakis and Ryan[32]. Furthermore, stochastic dominance bounds for Heston's[14] stochastic volatility model are obtained under certain assumptions. Finally, this thesis extends the work of Carr and Madan[5] and solves for the characteristic function of the call price given the physical characteristic function under the CRRA utility model.

Chapter 1

Tempered Stable Distribution

Throughout this thesis, various classes of stochastic processes will be used for examples. The purpose of this chapter is introduce these stochastic processes and make the reader familiar with them.

1.1 Levy Measures

A Levy Process is any stochastic process $\{X(t) : t > 0\}$ with the following properties:
1. $X(0) = 0$ with probability 1.
2. Independent increments: For $t < t' < t''$, $X(t'') - X(t')$ and $X(t') - X(t)$ are independent.
3. Stationarity: For $s, t \in \mathbb{R}^+$, $X(t) - X(s) \stackrel{d}{=} X(t - s)$.
Lawler[19] demonstrates that a Levy process has a characteristic function of the form $\mathbb{E}[e^{iuX(t)}] = e^{t\psi(u)}$, where the function ψ is called the characteristic exponent and takes the form

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}\setminus\{0\}} [e^{iux} - 1 - iux\chi_{|x|<1}(x)]M(dx),$$

where $\gamma \in \mathbb{R}$, $\sigma > 0$, and M is a measure that satisfies

$$\int_{\mathbb{R}\setminus\{0\}} (1 \wedge ||x||^2)M(dx) < \infty.$$

Every Levy processes can be characterized by the **Levy triplet** (γ, σ, M). The measure M is called a **Levy measure**. All Levy processes are also **infinitely divisible**. That is, for any $t > 0$, for any n, there exists a sequence of independent identically distributed (iid) random variables $\{X_{i,n}\}_{i=1}^{n}$ such that $X(t) = \sum_{i=1}^{n} X_{i,n}$.

1.2 Tilting and Tempering

Let the function f be a partial density function (PDF) with support on $\mathbb{R}_0^+$. f_θ is the tilted density of f if $f_\theta(x) = \frac{1}{L(\theta)} e^{-\theta x} f(x)$, where L is the Laplace transform of f. Taking the Laplace transform of the tilted density: $L_\theta(\lambda) := \int_{-\infty}^{\infty} e^{-\lambda x} e^{-\theta x} \frac{f(x)}{L(\theta)} dx = \frac{1}{L(\theta)} \int_{-\infty}^{\infty} e^{-(\lambda+\theta)x} f(x) dx = \frac{L(\theta+\lambda)}{L(\theta)}$. If f describes an infinitely divisible distribution with non-negative support, its Laplace transform take the form

$$L(\lambda) = \exp[\int_0^\infty (e^{-\lambda x} - 1) M(dx) - \lambda b]$$

for some Levy measure M and $b \geq 0$. Therefore,

$$L_\theta(\lambda) = \exp[\int_0^\infty (e^{(-\lambda-\theta)x} - 1) M(dx) - (\theta + \lambda) b] * \exp[\int_0^\infty (1 - e^{-\theta x}) M(dx) + \theta b]$$

$$= \exp[\int_0^\infty (e^{-\lambda x} - 1) e^{-\theta x} M(dx) - \lambda b] = \exp[\int_0^\infty (e^{-\lambda x} - 1) M_\theta(dx) - \lambda b],$$

where $M_\theta(dx) = e^{-\theta x} M(dx)$. M_θ is called the tilted Levy measure. Taking the product of convolution powers $f_{\theta_i}^{*r_i}$, where $\theta_i, r_i > 0$, one arrives at a Laplace transform of the form

$$\exp[\int_0^\infty (e^{-\lambda x} - 1) q(x) M(dx) - \lambda b],$$

where q is a completely monotone function with $\lim_{x\to\infty} q(x) = 0$. This is called **tempering**.

1.3 Tempered Stable Levy Measure

By Rosinski[30], an α-Stable distribution is a distribution equipped with the α-Stable Levy measure M_0, where

$$M_0(dx) = |x|^{-\alpha-1}dx,$$

where $\alpha \in (0,2)$. Then the **Tempered α-stable distribution** has Levy measure

$$M(dx) = q(x)M_0(dx) = |x|^{-\alpha-1}q(x)dx$$

for some $q : \mathbb{R} \to \mathbb{R}^+$ such that q is noin-increasing on $(0,\infty)$ and non-decreasing on $(-\infty,0)$ and $\lim_{|x|\to\infty} q(x) = 0$. The Levy measure M corresponds to a **proper** Tempered α-Stable distribution if $\lim_{x\to 0} q(x) = 1$. In Rosinski, it is shown that the Levy measure M of a Tempered Stable process is proper if and only if $\lim_{s\to 0^+} s^\alpha M(|x| > s) < \infty$. Let Q be some measure on $\mathcal{B}(\mathbb{R})$ such that

$$q(x) = \int_0^\infty e^{-|x|s}Q(ds)$$

Let R be some measure such that

$$R(A) = \int_{-\infty}^{\infty} \chi_A(x^{-1})|x|^\alpha Q(dx), \forall A \in \mathcal{B}(\mathbb{R}).$$

Then for any function $F : \mathbb{R} \to \mathbb{R}$,

$$\int_{-\infty}^{\infty} F(x)R(dx) = \int_{-\infty}^{\infty} F(x^{-1})|x|^\alpha Q(dx) \tag{1.1}$$

Therefore,

$$\begin{aligned} M(dx) =& q(x)|x|^{-\alpha-1}dx \\ =& \int_0^\infty e^{-|x|s}Q(ds)|x|^{-\alpha-1}dx \\ =& \int_0^\infty e^{-\frac{|x|}{s}} s^\alpha R(ds)|x|^{-\alpha-1}dx, \end{aligned}$$

where the third equality can be observed by taking $F(s) = e^{-\frac{|x|}{s}} s^{\alpha} \chi_{\mathbb{R}^+}(s)$ and plugging it into equation (1.1). R is called the **Rosinski measure**. It is shown in Rosinski that M is a Levy measure if and only if the Rosinski measure R satisfies $\int_{-\infty}^{\infty} |x|^2 \wedge |x|^{\alpha} R(dx) < \infty$ and $R(\{0\}) = 0$. It is also shown that the Levy measure M corresponds to a proper Tempered Stable Distribution if and only if $\int_{-\infty}^{\infty} |x|^{\alpha} R(dx) < \infty$.

1.4 The CGMY Distribution

The CGMY distribution has the spectral measure R of the form

$$R(dx) = \left(CG^Y \delta_{-\frac{1}{G}}(x) + CM^Y \delta_{-\frac{1}{M}}(x) \right) dx,$$

where $Y \in [0,2), C > 0, G > 0, M > 0$. If ν is some Levy measure such that

$$\nu(dx) = C\left(\frac{e^{-G|x|}}{|x|^{1+Y}} \chi_{x<0} + \frac{e^{-M|x|}}{|x|^{1+Y}} \chi_{x \geq 0}\right) dx$$

then the Levy triplet for a CGMY process is $(m, 0, \nu)$, where $m \in \mathbb{R}$ is some drift term. Thus the $CGMY$ distribution can be completely characterized by four parameters (plus a drift parameter). From Rachev[26], the characteristic exponent ψ then takes the form

$$\psi(u) = ium + C\Gamma(Y)(M - iu)^Y - M^Y + (G + iu)^Y - G^Y,$$

where m is the drift parameter.

1.4.1 Cumulants of the CGMY Distribution

The n^{th} moment μ_n of some random variable X is given by $\mu_n = \mathbb{E}[X^n]$. The n^{th} cumulant c_n is given by

$$c_n \sum_{k=1}^{n} (-1)^n (k-1)! B_{n,k}(\mu_1, ..., \mu_n),$$

where $B_{n,k}$ is the n^{th} k-incomplete Bell polynomial. From [26], the cumulants of the CGMY distribution can obtained directly from the characteristic function. The j^{th} cumulant c_j is given by

$$c_j = \Gamma(j-Y)\frac{C}{M^{j-Y}} + (-1)^j\Gamma(j-Y)\frac{C}{G^{j-Y}}$$

for $j \in \{1, 2, ...\}$. Note that this means the cumulants increases linearly with C.

1.5 The Variance Gamma Distribution

The Variance Gamma Distribution is a class of distributions with characteristic exponent ψ given by

$$\psi(u) = (1 - i\theta\kappa u + \frac{1}{2}\sigma^2\kappa u^2)^{-\frac{1}{\kappa}},$$

where $\theta \in \mathbb{R}, \sigma > 0, \kappa > 0$. If ν is some Levy measure such that

$$\nu(dx) = \frac{e^{\frac{\theta x}{\sigma^2}}}{\kappa|x|}\exp\left(-\frac{\sqrt{\frac{2}{\kappa}+\frac{\theta^2}{\sigma^2}}}{\sigma}|x|\right)dx$$

then the Levy triplet for the Variance Gamma distribution is given by $(\int_{|x|<1} x\nu(dx), 0, \nu)$. Observe that when $\theta = 0$, the variance gamma distribution is a CGMY distribution with $C = \frac{1}{\kappa}, G = M = \frac{\sqrt{\frac{2}{\kappa}}}{\sigma}, Y = 0$.

1.6 Analytic and Entire Characteristic Functions

In the complex plain, a function f is **analytic** on some set $A \subset \mathbb{C}$ if and only if there exists an open set U containing A such that $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ exists for all $z_0 \in U$. The function f is **entire** if and only if f is analytic on all of $\mathbb{C}$. The **order** α of an entire function f is given by

$$\alpha = \limsup_{r\to\infty}\frac{\ln(\ln ||f||_{\infty,B_r})}{\ln r},$$

where $||\cdot||_{\infty,B_r}$ denotes the infinity norm over the ball of radius r. The **type** τ of an entire function is given by

$$\tau = \limsup_{r\to\infty} \frac{\ln ||f||_{\infty,B_r}}{r}.$$

Future chapters will be dealing with theorems that assume an analytic characteristic function. This section will discuss the necessary and sufficient conditions for a random variable to have an analytic characteristic function. The following theorems provide useful properties of analytic characteristic functions[22]:

Theorem 1. *If F is a CDF then its characteristic function ψ is analytic on the strip $|\Im[z]| < R$ for some $R > 0$ if*

$$1 - F(x) + F(-x) = O(e^{-rx})$$

for any $0 < r < R$.

Intuitively, this is saying that the distribution must have sufficiently thin tails in order for its characteristic function to be analytic. One necessary condition is that all moments must be finite.

Theorem 2. *If F is a CDF then its characteristic function ψ is an entire function of order $1 + \alpha^{-1}$ and type τ if and only if*

$$\liminf_{x\to\infty} \frac{1}{x^{1+\alpha}\ln(1 - F(x) + F(-x))} = \frac{(\alpha\tau^{-1})^{\alpha}}{(1+\alpha)^{1+\alpha}}$$

and $1 - F(x) + F(-x) > 0$ for all $x > 0$.

1.7 Stochastic Differential Equations and Stochastic Integrals

A diffusive stochastic differential equation (SDE) takes the form:

$$\begin{cases} dS(t) = a(S(t),t)dt + b(S(t),t)dW(t) \\ S(0) = s_0 \end{cases}. \tag{1.2}$$

A Weiner process (also sometimes called a Brownian motion) W is a Levy process such that $W(t) - W(r) \sim \mathcal{N}(0, t-r) \forall t > r \geq 0$. The Stochastic Differential Equations (SDE) can be used to characterize a stochastic process in terms of a stochastic integral, which is the limit of Riemann sums using left endpoints. Under this definition of stochastic integration, (1.2) describes a process such that

$$
\begin{aligned}
S(t) =& s_0 + \int_0^t dS(t) \\
=& s_0 + \int_0^t a(S(t), t)dt + \int_0^t b(S(t), t)dW(t) \\
=& s_0 + \lim_{n\to\infty} \sum_{k=1}^{m_n} a(S(t_{k-1}^{(n)}), t_{k-1}^{(n)})(t_k^{(n)} - t_{k-1}^{(n)}) + \lim_{n\to\infty} \sum_{k=1}^{m_n} b(S(t_{k-1}^{(n)}), t_{k-1}^{(n)})(W(t_k^{(n)}) - W(t_{k-1}^{(n)})),
\end{aligned}
$$

where $0 = t_0^{(n)} < ... < t_{m_n}^{(n)} = t$ for every n, $\max_{1\leq k\leq m_n} |t_k^{(n)} - t_{k-1}^{(n)}| \to 0$ as $n \to \infty$, and W is a Wiener process. The above is the Ito definition of a stochastic integral. Ito calculus makes the following assumptions that are consistent with this representation of a stochastic integral. These assumptions are: $dW(t)dW(t) = dt$, $dW(t)dt = 0$, and $dt^2 = 0$. Using the Taylor expansion and the aforementioned assumptions, one arrives at Ito's Lemma:

Lemma 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let the stochastic process S be characterized by (1.2). Then*

$$
df(S(t), t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}a(S(t), t)dt + \frac{\partial^2 f}{\partial S^2}\frac{1}{2}b(S(t), t)^2 dt + \frac{\partial f}{\partial S}b(S(t), t)dW(t).
$$

Now, the diffusive SDE describes a process with almost surely continuous sample paths. If one wishes to characterize a process that has jump discontinuities, one can add a jump term characterized by a compound Poisson process. Then Jump-Diffusion SDE takes the form:

$$
\begin{cases}
dS(t) = a(S(t), t)dt + b(S(t), t)dW(t) + J_t dN_t \\
S(0) = s_0
\end{cases}
. \tag{1.3}
$$

Then (1.3) characterizes the process described by

$$
\begin{aligned}
S(t) =& s_0 + \int_0^t dS(t) \\
=& s_0 + \int_0^t a(S(t),t)dt + \int_0^t b(S(t),t)dW(t) + \int_0^t J_t dN(t) \\
=& s_0 + \lim_{n\to\infty} \sum_{k=1}^{m_n} a(S(t_k^{(n)}), t_k^{(n)})(t_k^{(n)} - t_{k-1}^{(n)}) + \lim_{n\to\infty} \sum_{k=1}^{m_n} b(S(t_k^{(n)}), t_k^{(n)})(W(t_k^{(n)}) - W(t_{k-1}^{(n)})) \\
&+ \lim_{n\to\infty} \sum_{k=1}^{m_n} J_{t_k^{(n)}} (N(t_k^{(n)}) - N(t_{k-1}^{(n)})),
\end{aligned}
$$

where J_t, N, and W are independent, $J_t \overset{d}{=} J$ for all t, where J is some random variable, and $N(t+s) - N(t) \sim \text{Poiss}(\lambda s)$ for some parameter λ. Ito's Lemma can also be extended to Jump-Diffusion processes as follows:

Lemma 2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let the stochastic process S be characterized by (1.2). Then*

$$
df(S(t),t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}a(S(t),t)dt + \frac{\partial^2 f}{\partial S^2}\frac{1}{2}b(S(t),t)^2 dt + \frac{\partial f}{\partial S}b(S(t),t)dW(t) + \left[f(S(t-)+J_t,t) - f(S(t-),t)\right] dN_t.
$$

Chapter 2

Introduction to Asset Pricing

Throughout this thesis, various arguments built on asset pricing theory will be used to derive formulas, theorems, and lemmas. This chapter introduces the foundation for asset pricing theory so that the reader may become familiar with the concepts and models used throughout this thesis. An **asset** or **security** can be described as an uncertain payment on some future date, which can be described by random variable or a stochastic process depending on the context. A **market** is a vector of assets. If M is a market and $(\Omega, \mathcal{F}, \mathbb{P})$ is some probability space such that M is $\mathcal{F}$ measurable then it is said that M is **defined on** $(\Omega, \mathcal{F}, \mathbb{P})$. A **portfolio** is a linear combination of assets. A **return** on some asset or portfolio over some fixed time interval is the change in the asset price. A portfolio is said to be **long** an asset if its return is increasing in the asset's return. A portfolio is said to be **short** an asset if its return is decreasing in the asset's return. A portfolio is said to be **self-financing** if it has a price of 0. A **replicating portfolio** of some security is a portfolio whose return is equal to the return on the security almost surely. A **riskless asset** is an asset whose return is deterministic. A probability measure $\mathbb{Q}$ is a **martingale measure** if all assets in the market are martingales under $\mathbb{Q}$. It is a **time-discounted martingale measure** if there exists some deterministic function of time $R : \mathbb{R}^+ \to \mathbb{R}$ such that any asset in the market multiplied by R is a martingale. The probability measures $\mathbb{P}$ and $\mathbb{Q}$ are **equivalent** on the σ-field $\mathcal{F}$ if, for any $A \in \mathcal{F}$, $\mathbb{P}[A] > 0$ if and only if $\mathbb{Q}[A] > 0$. Given a portfolio, **buying** x units of an asset adds x units of that asset to that portfolio. **Selling** x units of that asset subtracts x units from that portfolio. **Short selling** an asset is the act of selling the asset resulting in the portfolio being short the asset. The fundamental problem that asset pricing seeks to answer is as follows[7]: What is the value of a claim to an uncertain payment? The concept of arbitrage can be used to answer this question.

Definition 1. *Arbitrage is when a self-financing portfolio has, at some fixed date, an almost sure non-negative return, and positive probability of a positive return.*

If all investors are rational and prefer to have more money than less, then an investor will always take advantage of an opportunity for arbitrage. However, if investors keep buying an arbitrage portfolio, this will drive up the price of such a portfolio. Rational investors will drive up the price of the portfolio until the assets in the arbitrage portfolio become too expensive and the arbitrage opportunity disappears. Therefore, in equilibrium, assets should be priced in a way that allows for no arbitrage opportunities. This motivates the First Fundamental Theorem of Asset Pricing.

Theorem 3 (First Fundamental Theorem of Asset Pricing)**.** *Let a market be defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. No arbitrage opportunities in the market exist if and only if there exists a martingale measure* $\mathbb{Q}$ *equivalent to* $\mathbb{P}$ *such that every asset is priced according to its time-discounted expectation under* $\mathbb{Q}$*.*

Observe that since $\mathbb{Q}$ is equivalent to $\mathbb{P}$, there exists random variable D such that $\frac{d\mathbb{Q}}{d\mathbb{P}} =: D$, where D is $\mathcal{F}$-measurable, strictly positive and finite almost surely, and has expectation 1 (under $\mathbb{P}$). D is often referred to as the **stochastic discount factor**. The **consumption-based model** is a framework for asset pricing in which there exists a measurable random variable $U'(c)$ called **marginal utility of consumption** that describes the utility one derives from consuming a little more of the asset's payoff in the future. Under this framework, the stochastic discount factor is chosen to be the normalized marginal utility of consumption ($D = \frac{U'(c)}{E[U(c)]}$), which will be explained in greater detail in section (2.2). Of course, the marginal utility of consumption is high during bad times and low during good times (food tastes better when you're hungry).

2.1 Option Markets and Completeness

This section will state The Second Fundamental Theorem of Asset Pricing, which utilizes the concept of completeness:

Definition 2. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be the probability space on which a market is defined. The market is* ***complete*** *if any measurable random variable can be constructed from a portfolio of assets.*

The following a theorem provides an foundation for the basis of the Black-Scholes model, which will be expounded on in section 2.3:

Theorem 4 (Second Fundamental Theorem of Asset Pricing). *A market defined on the the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has no-arbitrage opportunities and is complete if and only if there exists a unique martingale measure $\mathbb{Q}$ such that every asset price is equal to the deterministically discounted expected price of that asset under $\mathbb{Q}$.*

In finance, an **option** gives the holder the option to buy or sell a particular asset, called the **underlying asset**, at a particular date or multiple particular dates. A European call option is an option that allows the holder to purchase an underlying asset on a given date, called the **expiry date**, for a predetermined price, called the **strike price**. If the price of the underlying asset falls below the strike price then the option is not exercised, since exercising the option would result in the owner of the option losing money. The price of an option can generally be described as a function of time and the underlying asset. In a complete market, a replicating portfolio always exists for an option. Therefore, in order to avoid an arbitrage opportunity, the price of the call must be equal to the price of the replicating portfolio. If the price of the call exceeds the price of the replicating portfolio, then one can purchase the replicating portfolio and short sell the call. The cost of the strategy will be negative. Then one can purchase a riskless bond to make up the difference and make the strategy self-financing. Such a strategy would pay out the return on the bond and would therefore statewise dominate 0. The opposite strategy statewise dominates 0 if the price of the replicating portfolio exceeds the price of the call.

2.2 Utility Functions and the Consumption-Based model

Rubinstein[31] introduces a model for the pricing of options, called the consumption-based model, which makes the following assumptions:

1. There is a single price for every asset.
2. All else equal, the greater the dividends of an asset, the greater the price of that asset.
3. There are no economies of scale and every participant in the market is a price-taker (i.e., they cannot noticeably influence the price). All investors can purchase the same security at the same date for the same price.
4. Every investor seeks to maximize a utility function over lifetime dollar consumption, which is concave and additive in each date.

5. There exists a representative investor such that

   - Every homogenous economic characteristic also describes the representative investor.
   - If a certain characteristic is measured in units of wealth, then this characteristic for the representative investor is the unweighted average over all investor characteristics.
   - Prices are determined as if all investors are average.

The assumption that the utility function is concave is essentially the same as the assumption that the representative investor is risk-averse, since concavity implies marginal utility diminishes as consumption increases. Now, if U is the utility function of the representative investor, C_t is the consumption of the representative investor at time t, S_t is the price of an asset at time t, and E_t is the consumption of the representative investor if they had not purchased the asset, then the above conditions imply that the representative investor wants to balance their portfolio in a way that maximizes total expected utility:

$$\max_{\xi} U(C_0) + \beta\mathbb{E}[U(C_t)]$$

$$s.t. \qquad C_t = E_t + \xi S_t$$

$$C_0 = E_0 - \xi S_0,$$

where β is some subjective time-discounting factor and t is the time until the next portfolio revision opportunity. First-order conditions yield $\frac{d}{d\xi}(U(C_0) + \beta\mathbb{E}[U(C_t)]) = \beta\mathbb{E}[S_t U'(C_t)] - S_0 U'(C_0) = 0$. Solving for S_0 yields $S_0 = \frac{\beta}{U'(C_0)}E[U'(C_t)S_t]$. Since the stochastic discount factor D must satisfy $E[D] = 1$, this allows for $D = \frac{U'(C_t)}{E[U'(C_t)]}$ and $\beta = \frac{U'(C_0)}{r\mathbb{E}[U'(C_t)]}$, where r is some constant so that $S_0 = r^{-1}\mathbb{E}[DS_t]$. Since U is concave, U' is decreasing in consumption, and thus so is D. This will be the framework used for asset pricing in all of the following sections.

2.3 The Black-Scholes Model

This section will use the information from the previous sections to derive and solve the famous Black-Scholes equation. First, assume there exists some riskless asset $b(t)$ described by the SDE

$$db(t) = rb(t)dt,$$

where r is some constant called the **instantaneous rate of return**. A U.S. treasury bond could be an example of such an asset. Black and Scholes[3] assumed that the change in the price of a stock $s(t)$ at time t could be described by the following diffusive SDE:

$$ds(t) = \mu s(t)dt + \sigma s(t)dw(t),$$

where $w(t)$ is a Weiner Process at time t. Under the Black-Scholes framework, the market is composed of only s and b and some option whose price $f(s(t), t)$ can be represented as a function of $s(t)$ and t. By Ito's Lemma, the change in price of a call option $f(s(t), t)$ on $s(t)$ at time t can be described by the following SDE:

$$df(s(t), t) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial s}\mu s(t) + \frac{\partial^2 f}{\partial s^2}\frac{1}{2}\sigma^2 s(t)^2 \right) dt + \frac{\partial f}{\partial s}\sigma s dw(t).$$

Now, construct a portfolio by buying a call and short selling $\frac{\partial f}{\partial s}$ units of the underlying stock. The change in the price of such a portfolio $p(s(t), t)$ at time t can be described by the following SDE:

$$dp(s(t), t) = df(s(t), t) - \frac{\partial f}{\partial s}ds = \left(\frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial s^2}\frac{1}{2}\sigma^2 s(t)^2 \right) dt.$$

Observe that this SDE has no random part. This means that, in order to avoid arbitrage, its return must equal the return of $\frac{p(s(t),t)}{b(t)}$ units of the riskless asset. Otherwise, if its return exceeded that of the riskless asset, it would be possible to construct a self-financing portfolio with almost sure positive return by buying the replicating portfolio and selling $\frac{p(s(t),t)}{b(t)}$ units of the riskless asset, resulting in arbitrage. So it must be the case that $dp(s(t), t) = \frac{p(s(t),t)}{b(t)} db(t) = rp(s(t), t)dt$. Since $p(s(t), t) = f(s(t), t) - \frac{\partial f}{\partial s} s$, this yields

$$\left(\frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial s^2}\frac{1}{2}\sigma^2 s(t)^2 \right) dt = \left(rf(s(t), t) - r\frac{\partial f}{\partial s} s \right) dt,$$

which implies

$$\frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial s^2}\frac{1}{2}\sigma^2 s(t)^2 - rf(s(t),t) + r\frac{\partial f}{\partial s}s = 0.$$

This is the famous Black-Scholes PDE. Boundary values can be obtained from the fact that a call option with strike price K pays out $f(s(T),T) = (s-K)^+$ at the expiration time T.
This PDE can be solved by taking using the Feynman-Kac Theorem. The closed form solution is

$$f(s(t),t) = \Phi(d_1)s(t) + \Phi(d_2)Ke^{-r(T-t)},$$

where $d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln(\frac{s(t)}{K}) + (r+\frac{\sigma^2}{2})(T-t)\right]$, $d_2 = d_1 - \sigma\sqrt{T-t}$, and Φ is standard normal CDF.

2.4 The Merton Model

Merton[23] argues that stock price returns generally do not have continuous sample paths and that "there is a prima facie case for the existence of jumps." Mertons assumes the following model for stock price movement:

$$ds_t = (\mu - k\lambda)s_t dt + \sigma s_t dw + (Y-1)s_t dN,$$

where dN is a poisson process differential independent of dw with jump intensity λ and Y is a log-normally distributed random variable with mean $k+1$. If $f(s(t),t)$ represents the price of a call at time t with underlying price $s(t)$ then, by Ito's lemma for Jump-Diffusion processes, it must satisfy the following SDE:

$$df(s(t),t) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial s}(\mu - \lambda k)s(t) + \frac{\partial^2 f}{\partial s^2}\frac{1}{2}\sigma^2 s(t)^2\right)dt + \frac{\partial f}{\partial s}\sigma s dw(t) + \left[f((Y-1)s-,t) - f(s-,t)\right]dN.$$

If a portfolio p is constructed by buying one call and selling $\frac{\partial f}{\partial s}$ shares as in the previous section, the following SDE must hold:

$$dp = df - \frac{\partial f}{\partial s} ds = \left(\frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial s^2} \frac{1}{2} \sigma^2 s(t)^2 \right) dt + \left(f((Y-1)s(t-), t) - f(s(t-), t) - \frac{\partial f}{\partial s}(Y-1)s(t) \right) dN.$$

The above SDE describes a pure jump process. Intuitively, such a portfolio would have a predictable return most of the time, but occasionally experience unpredictable jumps. Since, this portfolio is not riskless, arbitrage arguments cannot be made about its required return. To overcome this hurdle, Mertons assumed that the jumps were "non-systematic". That is, it is assumed the jumps the same expectation under the physical measure $\mathbb{P}$ as under the risk-neutral measure $\mathbb{Q}$. Taking the expectation, one arrives at

$$\mathbb{E}[dp] = \left(\frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial s^2} \frac{1}{2} \sigma^2 s(t)^2 - (\mathbb{E}[f((Y-1)s(t-), t) - f(s(t-), t)] - \frac{\partial f}{\partial s} ks(t))\lambda \right) dt = rpdt = rfdt - \frac{\partial f}{\partial s} sdt.$$

Rearranging yields

$$\frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial s^2} \frac{1}{2} \sigma^2 s^2 - \lambda \mathbb{E}[f((Y-1)s, t) - f(s, t)] - \frac{\partial f}{\partial s}(r - k\lambda)s - rf = 0.$$

If f is a call then the following boundary condition must hold $f(s(T), T) = (s(T) - K)^+$ for some fixed K. The solution to the PDE can be written in the form

$$f(s(t), t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)} \lambda^n (T-t)^n}{n} \mathbb{E}[f_{BS}(s(t) X_n e^{-\lambda k(T-t)}), t],$$

where f_{BS} is the Black-Scholes equation and the random variables $X_1, X_2, ...$ are independent, with $Y \stackrel{d}{=} X_1 \stackrel{d}{=} X_2 \stackrel{d}{=} ....$ In the case where Y is log-normally distributed, the price can be given in closed-form by

$$f(s(t), t) = \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)} \lambda^n (T-t)^n}{n} \mathbb{E}[f_n(s(t), t)],$$

where

$$f_n = f_{BS}(s(t), t; \sigma^2 + \frac{n}{T}\delta^2),$$

where the third argument is the variance parameter and δ is the variance of Y.

2.5 The Heston Model

In the special case of the Heston model, the stock price is described by the following system of SDEs:

$$dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}dz_2(t) \tag{2.1}$$

$$ds(t) = \mu s dt + \sqrt{v(t)}s dz_1(t) \tag{2.2}$$

where z_1 and z_2 are Wiener processes with constant correlation ρ. So, by Ito Calculus, $dz_1 dz_2 = \rho dt$, $dt^2 = 0$, $dz_i dt = 0$, $i = 1, 2$, and $dz_i^2 = dt$, $i = 1, 2$. Heston[14] uses this fact to derive the following Taylor Expansion for any portfolio P that is long one unit of a financial derivative C dependent on the stock price, volatility, and time, and short $\frac{\partial C}{\partial s}$ units of the stock:

$$dP = dC(s(t), v(t), t) - \frac{\partial C}{\partial s}ds = \left(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial v}\kappa(\theta - v) + \frac{1}{2}\frac{\partial^2 C}{\partial s^2} + \frac{\partial^2 C}{\partial s \partial v}\rho\sigma v s + \frac{1}{2}\frac{\partial^2 C}{\partial v^2}\sigma v\right)dt + \frac{\partial C}{\partial v}\sigma\sqrt{v}dz_2. \tag{2.3}$$

Now, observe that if r is the riskless rate then $e^{rdt} = 1 + rdt$. Also observe that $U'(c+dc) = U'(c) - \gamma U'(c)\frac{dc}{c} + O(dc^2) = U'(c)(1 - \gamma\frac{dc}{c}) + O(dc^2)$, where $\gamma = -\frac{cU''(c)}{U'(c)}$ is the relative risk aversion. If $U'(c)$ is the marginal utility of consumption and β is the subjective discount factor of the representative investor then the consumption-based model implies:

$$
\begin{aligned}
PU'(c) =&\beta E[(P+dP)U'(c+dc)] \\
=&\beta E[U'(c+dc)]P + \beta E[U'(c+dc)]\left(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial v}\kappa(\theta - v) + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}vs^2 + \frac{\partial^2 C}{\partial s \partial v}\rho\sigma v s + \frac{1}{2}\frac{\partial^2 C}{\partial v^2}\sigma^2 v\right) dt \\
&+ \frac{\partial C}{\partial v}\sigma\sqrt{v}\beta E[dz_2 U'(c+dc)] \\
=&(1-rdt)U'(c)P + (1-rdt)U'(c)\left(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial v}\kappa(\theta - v) + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}vs^2 + \frac{\partial^2 C}{\partial s \partial v}\rho\sigma v s + \frac{1}{2}\frac{\partial^2 C}{\partial v^2}\sigma^2 v\right) dt \\
&+ \frac{\partial C}{\partial v}\sigma\sqrt{v}(-U'(c)\gamma\beta E[dz_2\frac{dc}{c}] + \beta E[dz_2 O(dc^2)]).
\end{aligned}
$$

Heston assumes that γ is constant and that consumption c is driven by the following process:

$$dc = \mu_c c dt + \sigma_c \sqrt{v} c dz_3,$$

where μ_c and σ_c are constants and dz_3 is some Weiner process (that may be correlated with dz_2 and dz_1). Thus the higher order terms are constant (and cancel with the expectation). Using these facts and rearranging the above equation gives us the following PDE:

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial v}(\kappa(\theta - v) - \lambda v) + \frac{1}{2}\frac{\partial^2 C}{\partial s^2} + \frac{\partial^2 C}{\partial s \partial v}\rho\sigma v s + \frac{1}{2}\frac{\partial^2 C}{\partial v^2}\sigma^2 v + r\frac{\partial C}{\partial s}s - rC = 0,$$

where λ is some constant. The Heston model has some difficulties:

- The assumption that λ is constant relies on consumption being driven by the above diffusive process and on the relative risk aversion γ being constant, neither of which are trivial or obvious assumptions.
- The estimation of λ requires us to know the price of another volatility-dependent asset.

In chapter 5, bounds are estimated that rely on less stringent assumptions and do not rely on parameters that need to be estimated from another volatility-dependent asset.

2.6 Parameter Estimation

2.6.1 Method of Moments

One can obtain moments from physical stock price behavior. Furthermore, one
can discretize over time by letting $Q_t := \frac{S_{t+1}}{S_t}$ and modeling the dynamics of $\{Q_t\}_{t=1}^{n}$ by

$$Q_{t+1} = 1 + \mu + \sqrt{V_t}(\rho Z_1 + \sqrt{1-\rho^2} Z_2)$$

$$V_{t+1} = V_t + \kappa(\theta - V_t) + \sigma\sqrt{V_t} Z_1$$

where Z_1, Z_2 are independent standard normal variables.One can estimate the j^{th} empirical moment μ_j by $\frac{1}{n}\sum_{t=1}^{n} Q_t^j := \mu_j$. Dunn et al.[11] gives the following expression for the first, second, fourth and fifth moments in terms of the model parameters:

$$\begin{aligned}
\mu_1 =& 1 + \mu \\
\mu_2 =& (1+\mu)^2 + \theta \\
\mu_4 =& \frac{1}{\kappa(\kappa-2)}(\kappa^2\mu^4 + 4\kappa^2\mu^3 + 6\mu^2\kappa^2\theta - 2\kappa\mu^4 + 6\kappa^2\mu^2 + 12\kappa^2\mu\theta + 3\kappa^3\theta^3 - 8\kappa\mu^3 \\
& - 12\kappa\mu^2\theta + 4\kappa^2\mu + 6\kappa^2\theta - 12\kappa\mu^2 - 24\kappa\mu\theta - 6\kappa\mu\theta^2 - 3\sigma^3\theta + \kappa^2 - 8\kappa\mu - 12\kappa\theta - 2\kappa) \\
\mu_5 =& \frac{1}{\kappa(\kappa-2)}(\kappa^2\mu^5 + 5\kappa^2\mu^4 + 10\mu^2\kappa^3\theta - 2\kappa\mu^5 + 10\kappa^2\mu^3 + 30\kappa^2\mu^2\theta + 15\kappa^2\mu\theta^2 - 10\kappa\mu^4 \\
& - 20\kappa\mu^3\theta + 10\kappa^2\mu^2 + 30\kappa^2\theta\mu + 15\kappa^2\mu^2 - 20\kappa\mu\theta + 10\kappa^2\mu^2 - 30\kappa^2\mu\theta + 15\kappa^2\theta^2 - 20\kappa\mu^3 - 60\kappa\mu^2\theta \\
& - 30\kappa\mu\theta^2 - 15\mu\sigma^2\theta + 5\kappa^2\mu + 10\kappa^2\theta - 20\kappa\mu^2 - 60\kappa\mu\theta - 30\kappa\theta^2 - 15\sigma^2\theta + \kappa^2 - 10\kappa\mu - 20\kappa\theta - 2\kappa)
\end{aligned}$$

Note that this is a system of 4 equations and 4 unknowns, as ρ is missing. This is one shortcoming of the method of moments. Furthermore, if $\phi_H(\cdot;\mu,\kappa,\theta,\sigma,\rho) =: \phi$ is the characteristic function of the log-stock price at $t = 1$ then it follows that $\mu_n = S_0^{-n}\phi(-in)$ for any natural number n. Therefore, the method of moments is choosing parameters so that $\phi(-i) = S_0\mu_1, \phi(-2i) = S_0^2\mu_2, \phi(-4i) = S_0^4\mu_4$, and $\phi(-5i) = S_0^5\mu_5$.

2.6.2 Maximum Likelihood Estimate (MLE)

The maximum likelihood estimate for the parameters is the choice of parameters so that
$\ell(\mu,\kappa,\theta,\sigma,\rho) := \sum_{t=1}^{n-1} \log(f(Q_{t+1}, V_{t+1};\mu,\kappa,\theta,\sigma,\rho))$ is maximized, where f is the joint density of Q and V. The advantage of this method is it generates all parameters. However, the disadvantage of this method is it requires us to estimate $\{V_t\}_{t=1}^{n}$. Since $Q_{t+1} \sim \mathcal{N}(1+\mu, V_t)$ and $V_t \sim \mathcal{N}(V_t - \kappa(\theta - V_t), \sigma^2 V_t)$ and $\rho = \text{corr}(Q_{t+1}, V_{t+1})$, the joint density f is given by

$$\begin{aligned} f(Q_{t+1}, V_{t+1};\mu,\kappa,\theta,\sigma,\rho) =& \frac{1}{2\pi\sigma V_t\sqrt{1-\rho^2}} \exp[-\frac{(Q_{t+1}-1-\mu)^2}{(1-\rho^2)} \\ &+ \frac{\rho(Q_{t+1}-1-\mu)(V_{t+1}-V_t-\theta\kappa+\kappa V_t)}{V_t\sigma\rho} \\ &- \frac{(V_{t+1}-V_t-\theta\kappa+\kappa V_t)^2}{2V_t\sigma^2\rho}] \end{aligned}$$

Therefore, the likelihood function is given by

$$\begin{aligned} \ell(\mu,\kappa,\theta,\sigma,\rho) = \sum_{t=1}^{n-1}&[\log(2) - \log(\pi) - \log(\sigma) - \log(V_t) - \log(\sqrt{1-\rho^2}) - \frac{(Q_{t+1}-1-\mu)^2}{(1-\rho^2)} \\ &+ \frac{\rho(Q_{t+1}-1-\mu)(V_{t+1}-V_t-\theta\kappa+\kappa V_t)}{V_t\sigma\rho} \\ &- \frac{(V_{t+1}-V_t-\theta\kappa+\kappa V_t)^2}{2V_t\sigma^2\rho}]. \end{aligned}$$

2.7 Implementation

Heston[14] found the a closed-form solution to the PDE by assuming the call price H takes the form

$$H(s(t), v(t), t; \kappa, \theta, \rho, \sigma, k, T, r, \lambda) = sP_1 - e^{-rT}kP_2$$

as in the Black-Scholes equation. Heston demonstrated that P_1 and P_2 are probability distributions that satisfy

$$\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2} + \rho\sigma v\frac{\partial^2 P_j}{\partial x\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 P_j}{\partial v^2} + (r+u_j v)\frac{\partial P_j}{\partial x} + (k\theta - b_j)\frac{\partial P_j}{\partial v} + \frac{\partial P_j}{\partial t} = 0,$$

where $u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, b_1 = \kappa + \lambda - \rho\sigma, b_2 = \kappa + \lambda, x = \ln s$, with terminal condition $P_j(x, v, T; k) = \chi_{[x \leq k]}$. Now let f_1 and f_2 satisfy the same PDE corresponding to P_1 and P_2, respectively, but with the boundary condition $f_j(x, v, T; u) = e^{iux}$. The solutions f_1 and f_2 are the characteristic functions of P_1 and P_2 since the expectation of the terminal price under the $\mathbb{Q}$ measure $E^{\mathbb{Q}}[e^{iuX}]$ is the Fourier transform by definition. Heston found that

$$f_j(x, v, t; u) = \exp[C(T-t; u) + D(T-t; u)v + iux],$$

where

$$C(\tau; u) = ir\tau + \frac{k\theta}{\sigma^2}[(b_j - \rho\sigma ui + d)\tau - 2\ln(\frac{1 - ge^{d\tau}}{1-g})],$$

$$D(\tau; u) = \frac{b_j - \rho\sigma ui + d}{\sigma^2}(\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}),$$

$$g = \frac{b_j - \rho\sigma ui + d}{b_j - \rho\sigma ui - d},$$

$$d = \sqrt{(\rho\sigma ui - b_j^2) - \sigma^2(2u_j ui - u^2)}.$$

Applying the inverse Fourier Transform for CDFs to obtain the P_j's:

$$P_j[x, v, t; \ln k] = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re[\frac{e^{iu\ln k} f(x, v, t; u)}{iu}]du.$$

Numerically, the P_j's can be obtain using a Fast Fourier Transform algorithm. The script for generating figure (5.2) can be found in Appendix section (8.4).

2.8 The CRRA Model

Constant Relative Risk Aversion describes a state of nature where the representative investor's risk preference relative to their consumption is independent with their level of consumption. The relative risk aversion $R(c)$ is the negative of the relative change in marginal utility over the relative change in consumption:

$$R(c) = -\frac{dU'(c)/U'(c)}{dc/c} = -c\frac{U''(c)}{U'(c)}.$$

When R is some constant γ, the following function solves the above differential equation:

$$U(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}$$

for $\gamma \neq 1, \gamma > 0$. Moreover, it follows from l'Hospital's rule that $\lim_{\gamma \to 1} U(c) = \ln c$. Therefore, it is common to use

$$U(c) = \begin{cases} \frac{c^{1-\gamma}-1}{1-\gamma} & \gamma > 0, \gamma \neq 1 \\ \ln c & \gamma = 1 \end{cases}.$$

The above class of functions is called the isoelastic utility functions, and the parameter γ is referred to simply as the relative risk aversion. Under such a model of utility, if q is the risk neutral density of an index then $q(x) = Nx^{-\gamma} f(x)$, where f is the empirical density and N is some normalization constant chosen so that $\int_0^\infty q(x)dx = 1$. An efficient algorithm for computing the call price under the CRRA model will be explored in chapter 6.

2.9 Fast Fourier Transform

The FFT algorithm is an efficient method for computing the following sum:

$$\sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j), \qquad k = 1, ..., N.$$

Assume N is even and factors into rs and let $j = rj_1 + j_2$, where $j_1 = 0, 1, ..., s-1$ and $j_2 = 1, 2, ..., r$. Similarly, let $k = sk_1 + k_2$, where $k_1 = 0, 1, ..., r-1$ and $k_2 = 1, ..., s$. Then

$$\begin{aligned}
\sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)}x(j) &= \sum_{j_2=1}^{r} e^{-i\frac{2\pi}{N}(j_2-1)(k-1)} \sum_{j_2=0}^{s-1} e^{-i\frac{2\pi}{N}(rj_1)(k-1)}x(rj_1+j_2) \\
&= \sum_{j_2=1}^{r} e^{-i\frac{2\pi}{N}(j_2-1)(k-1)} \sum_{j_2=0}^{s-1} e^{-i\frac{2\pi}{N}(rj_1)(sk_1)}e^{-i\frac{2\pi}{N}(rj_1)(k_2-1)}x(rj_1+j_2) \\
&= \sum_{j_2=1}^{r} e^{-i\frac{2\pi}{N}(j_2-1)(k-1)} \sum_{j_2=0}^{s-1} e^{-i2\pi j_1 sk_1}e^{-i\frac{2\pi}{N}(rj_1)(k_2-1)}x(rj_1+j_2) \\
&= \sum_{j_2=1}^{r} e^{-i\frac{2\pi}{N}(j_2-1)(k-1)} \sum_{j_2=0}^{s-1} e^{-i\frac{2\pi}{N}(rj_1)(k_2-1)}x(rj_1+j_2) \\
&= \sum_{j_2=1}^{r} e^{-i\frac{2\pi}{N}(j_2-1)(k_1)} \sum_{j_2=0}^{s-1} e^{-i\frac{2\pi}{N}(rj_1+j_2-1)(k_2-1)}x(rj_1+j_2)
\end{aligned}$$

Observe that for the inner sum of the last line, there is one multiplication and addition for each of the s different values of j_1 for each of the r different values of j_2 for each of the s different values of k_2. Then, for the outer sum of the last line, there is one multiplication and addition for each of the r different values of j_2 for each of the r different values of k_1 for each of the s different values of k_2. Consequently, the total computational time is of order $rs^2 + sr^2 = N(r+s)$. Compare this with the original formulation of the sum, wherein one would have to compute a multiplication and addition for each of the N different values of j for each of the N different values of k, resulting in a computational time of order N^2. This is a significant speed boost. For example, consider the case where $N = 2^p$. Then N can be broken down into p factors of 2, resulting in a computational time of order $N \log_2 N$. For this reason, it is desirable to use FFT to compute call prices. This will be significant for discussing efficient computation of call prices in section 4.1.1 and chapter 6.

2.10 Carr and Madan

This section reviews a well-known application of fft in option pricing, extending the ideas discussed in section 2.9. Carr and Madan[5] solve for the Fourier Transform of a call in terms of the Fourier Transform of the risk-neutral distribution. Carr and Madan finds that, for $\alpha > 0$ chosen so that the $\alpha + 1$ statistical moment of the underlying is finite, the Fourier Transform of a call option $f(e^k)$ at strike price e^k multiplied by the factor $e^{\alpha k}$ is given by $\psi(u) = e^{-rT}\frac{\phi(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+(2\alpha+1)ui}$, where r is the instantaneous riskless rate, T is the time to expiry, and ϕ is the Fourier Transform of the risk-neutral distribution. Therefore, the price of a call at strike price e^k is given by

$$\frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\psi(u)du$$

The difficulty here is the risk-neutral characteristic function is required as an input, which isn't always easy to obtain. In practice, it is almost always easier to first obtain the physical characteristic function and then assume some functional form for the pricing kernel (usually a power function, as CRRA is by far the most common model for utility used in the literature). Chapter 6 will explore an efficient algorithm for pricing options when only a physical characteristic function and a CRRA pricing kernel are given.

Chapter 3

Introduction to Stochastic Dominance Asset Pricing

The previous chapter dealt with more general and foundational concepts in asset pricing. This section will review a more specialized subtopic within the larger body of literature. Stochastic dominance will be used to justify both previous and original discoveries regarding asset price bounds throughout this thesis.

3.1 Stochastic Dominance

Stochastic Dominance is a term that describes a class of partial orderings on random variables. This section will discuss three stochastic dominance orderings in particular: statewise dominance, first-order dominance, and second-order dominance.

Definition 3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If X and Y are measurable random variables then X is said to* ***Statewise Dominate*** *Y if $X \geq Y$ almost surely and $X > Y$ with non-zero probability.*

Definition 4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If X and Y are measurable random variables then X is said to* ***First-Order Dominate*** *Y if $E[U(X)] \geq E[U(Y)]$ for all non-decreasing functions U and $E[U(X)] > E[U(Y)]$ for some non-decreasing function U.*

Definition 5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If X and Y are measurable random variables then X is said to* ***Second-Order Dominate***

Y if $E[U(X)] \geq E[U(Y)]$ for all non-decreasing concave functions U and $E[U(X)] > E[U(Y)]$ for some non-decreasing concave function U.

In finance, the term arbitrage is often used to describe a scenario in which self-financing trading strategy that statewise dominates 0. In the following sections, "second-order stochastic dominance bounds" will refer to bounds that rely on the assumption that investors behave in a way that maximizes some utility function with an increasing and concave utility function. That is, if one portfolio second-order dominates another, then the former portfolio will cost more than the latter. The term **second-order stochastic dominance arbitrage** to describes a scenario in which a self financing trading strategy second-order dominates another self-financing trading strategy.

3.2 Lattice Pricing

In the context of the asset pricing, a lattice can be thought of as the possible prices for an asset after a given period. In particular, a lattice is used to discretize the distribution of prices on a particular asset with a particular time to maturity.

3.2.1 The Binomial Lattice

A binomial lattice is a lattice with only two outcomes. Consider a stock whose price S after one period can be only one of two values: S_u and S_d. It's price process can be described in discrete time by a binomial lattice. In a complete market, the price of any call C can be replicated by a portfolio containing the underlying asset and a riskless bond. Let C_u represent the price of the call if $S = S_u$ and let C_d represent the price of the call if $S = S_d$. Now consider a and b chosen so that

$$\begin{cases} aS_u - b(1+r) = C_u \\ aS_d - b(1+r) = C_d \end{cases},$$

where r is the riskless rate. The above system of equations has a unique solution for a and b. Therefore, there exists a replicating portfolio $aS + b$ for C wherein one buys a units of the stock and burrows \$$b$ at the riskless rate. Observe that the system of equations has a unique solution if and only if there are two possible values for S. Introducing a third outcome (and thus a third linearly independent equation) would make the system unsolvable.

3.2.2 The Multinomial Lattice

Unfortunately, markets do not tend to be complete. To capture this limitation in discrete-time, Ritchken models the potential outcomes with a multinomial lattice (with greater than two outcomes). In practice, this implies the representative investor cannot make revisions to their portfolio arbitrarily quickly. In a single-period, the multinomial lattice gives n possible outcomes $\{s_1, ..., s_n\}$ for the price of the stock S and n possible outcomes $\{c_1, ..., c_n\}$ the call C. Though a unique no-arbitrage price on the price of the call can no longer be derived, no-arbitrage bounds on the call price can be derived.

3.3 Ritchken's Bounds

3.3.1 No-arbitrage bounds

Ritchken[28] considered a discrete-time, discrete-space scenario, starting with one-period period in discrete time: Let $\{s_1, ..., s_n\}$ represent the possible stock prices after one period, where $s_1 < s_2 < ... < s_n$. and let $\{c_1, ..., c_n\}$ be their corresponding prices on a call on the stock with strike price x. The fundamental theorem of calculus states there exist risk neutral probabilities $\{q_1, ..., q_n\}$ and some constant factor r such that $r^{-1} \sum_i c_i q_i = c$ and $r^{-1} \sum_i s_i q_i = s$, where s is the current price of the stock, and c is the current price of a call. Now let $\{e_1, ..., e_n\}$ be a set of constants such that $e_i := r^{-1} q_i$. Then finding bounds on c is a matter of solving the following optimization problems with the same constraints:

$$\max_{\{e_i\}} c \qquad \min_{\{e_i\}} c$$

subject to

$$c = \sum_i c_i e_i$$

$$\sum_i e_i = r^{-1},$$

$$\sum_i s_i e_i = s,$$

$$e_i \geq 0 \qquad \forall i.$$

The above optimization problems are what are known as linear programming problems; their so-called dual problems are, respectively:

$$\min_{y_1, y_2} c \qquad \max_{y_1, y_2} c$$

subject to

$$c = y_1 s + y_2 r^{-1}$$

$$y_1 s_i + y_2 \geq c_i, \forall i \qquad y_1 s_i + y_2 \leq c_i, \forall i.$$

The primal and dual problems have the some optimal solution for c. For a given value of c, if there is a choice of $\{e_i\}$ and y_1 and y_2 such that the former satisfies the constraints of the maximization (resp. minimization) primal problem and the latter satisfies the constraints of the dual minimization (maximization) problem then it follows that the optimal solution to the primal maximization (minimization) problem is greater (less) than c. However, the optimal solution to the dual minimization (maximization) problem is less (greater) than c. Therefore, since the optimal solution to both problems are the same, c must be the optimal solution. Therefore, to find an optimal solution, it suffices to find a c in the feasible region of both the primal and dual problem. Let

$$e_j = \begin{cases} \frac{s_{j^*+1} r^{-1} - s}{s_{j^*+1} - s_{j^*}}, & j = j^* \\ \frac{s - s_{j^*} r^{-1}}{s_{j^*+1} - s_{j^*}}, & j = j^* + 1 \\ 0 & \text{otherwise} \end{cases}, \qquad \forall j,$$

where $j^* = \arg\max\{s_{j^*} \leq rs\}$. It satisfies the constraints of the primal problem since $\sum_j e_j = \frac{s_{j^*+1} r^{-1} - s}{s_{j^*+1} - s_{j^*}} + \frac{s - s_{j^*} r^{-1}}{s_{j^*+1} - s_{j^*}} = r^{-1}$ and $\sum_j e_j s_j =$

$\frac{s_{j^*+1}r^{-1}-s}{s_{j^*+1}-s_{j^*}}s_{j^*} + \frac{s-s_{j^*}r^{-1}}{s_{j^*+1}-s_{j^*}}s_{j^*+1} = s$. Furthermore,

$$\begin{aligned} c =& \frac{s_{j^*+1}r^{-1} - s}{s_{j^*+1} - s_{j^*}}c_{j^*} + \frac{s - s_{j^*}r^{-1}}{s_{j^*+1} - s_{j^*}}c_{j^*+1} \\ =& \frac{c_{j^*}s_{j^*+1} - c_{j^*+1}s_{j^*}}{s_{j^*+1} - s_{j^*}}r^{-1} + \frac{c_{j^*+1} - c_{j^*}}{s_{j^*+1} - s_{j^*}}s. \end{aligned}$$

Now let $y_1 = \frac{c_{j^*+1}-c_{j^*}}{s_{j^*+1}-s_{j^*}}, y_2 = \frac{c_{j^*}s_{j^*+1}-c_{j^*+1}s_{j^*}}{s_{j^*+1}-s_{j^*}}$ and let $R(s_i, s_j) = \frac{c_i-c_j}{s_i-s_j}$. Since a call is convex in the stock price, by definition of convexity, $c_{x_2} \leq tc_{x_1} + (1-t)c_{x_3}$ for $x_3 \geq x_2 \geq x_1$ and t chosen so that $s_{x_2} = ts_{x_1} + (1-t)s_{x_3}$. Rearranging the former inequality, one arrives at $t(c_{x_3} - c_{x_1}) \leq c_{x_3} - c_{x_2}$. Dividing both sides by $s_{x_3} - s_{x_2} = t(s_{x_3} - s_{x_1})$, one arrives at $R(s_{x_1}, s_{x_3}) = \frac{c_{x_3}-c_{x_2}}{s_{x_3}-s_{x_2}} \geq \frac{c_{x_3}-c_{x_1}}{s_{x_3}-s_{x_1}} = R(s_{x_1}, s_{x_2})$. Furthermore, rearranging the former equation differently, one also arrives at $c_{x_1} - c_{x_2} \geq (1-t)(c_{x_1} - c_{x_3})$. Diving both sides by $s_{x_1} - s_{x_2} = (1-t)(s_{x_1} - s_{x_3})$, one arrives at $R(s_{x_2}, s_{x_1}) = \frac{c_{x_1}-c_{x_2}}{s_{x_1}-s_{x_2}} \leq \frac{c_{x_1}-c_{x_3}}{s_{x_1}-s_{x_3}} = R(s_{x_3}, s_{x_1})$. $R(s_i, s_j)$ is non-decreasing in s_i for fixed s_j when $s_j \leq s_i$ or when $s_j \leq s_i$. Therefore $\frac{c_k-c_{j^*}}{s_k-s_{j^*}} = R(s_k, s_{j^*}) \leq R(s_{j^*+1}, s_{j^*}) = \frac{c_{j^*+1}-c_{j^*}}{s_{j^*+1}-s_{j^*}} = y_1$ for $k \leq j^*$ and $\frac{c_k-c_{j^*}}{s_k-s_{j^*}} = R(s_k, s_{j^*}) \geq R(s_{j^*+1}, s_{j^*}) = \frac{c_{j^*+1}-c_{j^*}}{s_{j^*+1}-s_{j^*}} = y_1$ for $k > j^*$. The above inequalities imply $c_k \geq c_{j^*} + y_1 s_k - \frac{c_{j^*+1}s_{j^*}-c_{j^*}s_{j^*}}{s_{j^*+1}-s_{j^*}} = y_1 s_k + \frac{c_{j^*}s_{j^*+1}-c_{j^*+1}s_{j^*}}{s_{j^*+1}-s_{j^*}} = y_1 s_k + y_2$ for all k. So the minimization problem is solved and $c = \frac{s_{j^*+1}r^{-1}-s}{s_{j^*+1}-s_{j^*}}c_{j^*} + \frac{s-s_{j^*}r^{-1}}{s_{j^*+1}-s_{j^*}}c_{j^*+1}$ is a lower bound. Now let

$$e_j = \begin{cases} \frac{s_n r^{-1}-s}{s_n-s_1}, & j = 1 \\ \frac{s-s_1 r^{-1}}{s_n-s_1}, & j = n \\ 0 & \text{otherwise} \end{cases}, \qquad \forall j.$$

Clearly this choice of e_j's satisfies the primal problem's constraints. Moreover,

$$c = \frac{s_n r^{-1} - s}{s_n - s_1}c_1 + \frac{s - s_1 r^{-1}}{s_n - s_1}c_n \tag{3.1}$$

$$= \frac{c_1 s_n - c_n s_1}{s_n - s_1}r^{-1} + \frac{c_n - c_1}{s_n - s_1}s. \tag{3.2}$$

Now let $y_2 = \frac{c_1 s_n - c_n s_1}{s_n - s_1}, y_1 = \frac{c_n - c_1}{s_n - s_1}$. Observe that $\frac{c_k - c_1}{s_k - s_1} = R(s_k, s_1) \leq R(s_n, s_1) = \frac{c_n - c_1}{s_n - s_1} = y_1$ for all k. Then $c_k \leq c_1 + y_1 s_k - \frac{c_n s_1 - c_1 s_1}{s_n - s_1} = y_1 s_k + \frac{c_1 s_n - c_n s_1}{s_n - s_1} = y_1 s_k + y_2$. So the maximization problem is solved and $c = \frac{s_n r^{-1} - s}{s_n - s_1} c_1 + \frac{s - s_1 r^{-1}}{s_n - s_1} c_n$ is an upper bound.

3.3.2 Continuous Space No-arbitrage bounds

Now consider the price of a call option where the underlying stock has a continuous distribution. Assume the support for the price of the call in the next time step is contain in the interval $[s_{min}, s_{max}]$. Observe that in the discrete-space case, the upper bound $\overline{c}$ and lower bound $\underline{c}$ is

$$\overline{c} = \frac{s_n r^{-1} - s}{s_n - s_1} c_1 + \frac{s - s_1 r^{-1}}{s_n - s_1} c_n,$$

$$\underline{c} = \frac{s_{j^*+1} r^{-1} - s}{s_{j^*+1} - s_{j^*}} c_{j^*} + \frac{s - s_{j^*} r^{-1}}{s_{j^*+1} - s_{j^*}} c_{j^*+1},$$

where s_j denotes the j^{th} possible stock price. Now, fix $\alpha > 0$ and let $\{\mathcal{P}_\Delta\}_{\Delta \in (0,\alpha]}$ be a set of partitions such that $\mathcal{P}_\Delta = \{s_1, ..., s_{j^*(\Delta)}, s_{j^*(\Delta)+1}, ..., s_{n(\Delta)}\}$, where $\Delta = \max_{j \in \{1,...,n(\Delta)-1\}} |s_{j+1} - s_j|$, $s_1 = s_{min}$, and $s_{n(\Delta)} = s_{max}$. Bounds can be found for the continuous case by taking the limit of the bounds as $\Delta \to 0$. For any $N \in \mathbb{N}$, there exists $\epsilon > 0$ such that $\Delta < \epsilon$ implies $0 \leq r - \frac{s_{j^*(\Delta)}}{s} < 2^{-N}$ and $0 < \frac{s_{j^*(\Delta)+1}}{s} - r < 2^{-N}$. Now let $\delta = r - \frac{s_{j^*(\Delta)}}{s}$ and let $\delta' = r - \frac{s_{j^*(\Delta)}}{s}$. If the strike price x is less than sr and $2^{-N} < sr - x$ then

$$\begin{aligned}
\underline{c} =& \frac{s_{j^*(\Delta)+1} r^{-1} - s}{s_{j^*(\Delta)+1} - s_{j^*(\Delta)}} c_{j^*(\Delta)} + \frac{s - s_{j^*(\Delta)} r^{-1}}{s_{j^*(\Delta)+1} - s_{j^*(\Delta)}} c_{j^*(\Delta)+1} \\
=& \frac{r^{-1} s(r + \delta') - s}{s(1 + \delta') - s(1 - \delta)} (s(r - \delta) - x) + \frac{s - r^{-1} s(r - \delta)}{s(1 + \delta') - s(1 - \delta)} (s(r + \delta') - x) \\
=& \frac{r^{-1} \delta'}{\delta + \delta'} s(r - \delta) + \frac{r^{-1} \delta}{\delta + \delta'} s(r + \delta') - x r^{-1} \\
=& s - x r^{-1}.
\end{aligned}$$

Alternatively, if x exceeds sr then $\underline{c} = 0$ if $2^{-N} < x - sr$. Finally, if $x = sr$ then $\underline{c} = \frac{r^{-1}\delta}{\delta + \delta'}(s(r + \delta') - x) = \frac{r^{-1}\delta\delta'}{\delta + \delta'} s$. Observe that $0 \leq \frac{r^{-1}\delta\delta'}{\delta + \delta'} s \leq r^{-1} s 2^{-N}$. So, for general x, $\max(0, s - \frac{x}{r}) \leq \underline{c} \leq \max(0, s - \frac{x}{r}) + r^{-1} s 2^{-N}$. Since

N can be arbitrarily large $\lim_{\Delta \to 0} \underline{c} = \max(0, s - \frac{x}{r})$. However, $\overline{c}$ remains the same in the continuous case since $s_{n(\Delta)}$ and s_1 remain unchanged as Δ approaches 0. Also, if $s_1 = 0$ then $\overline{c} = s - \frac{xs}{s_{max}}$.

3.3.3 Discrete Time Bounds with Multiple Revision Opportunities

This section will use subscripts to denote time steps. If the representative investor has M revision opportunities then, for a fixed stock price $s^{(m)}$ at time step m, the price of a call $c^{(m)}$ at the m^{th} time step is

$$\frac{s_n^{(m+1)} r^{-1} - s^{(m)}}{s_n^{(m+1)} - s_1^{(m+1)}} c_1^{(m+1)} + \frac{s^{(m)} - s_1^{(m+1)} r^{-1}}{s_n^{(m+1)} - s_1^{(m+1)}} c_n^{(m+1)} \geq c \geq \frac{s_{j^*+1}^{(m+1)} r^{-1} - s^{(m)}}{s_{j^*+1}^{(m+1)} - s_{j^*}^{(m+1)}} c_{j^*}^{(m+1)} + \frac{s^{(m)} - s_{j^*}^{(m+1)} r^{-1}}{s_{j^*+1}^{(m+1)} - s_{j^*}^{(m+1)}} c_{j^*+1}^{(m+1)}.$$

However, observe that $\frac{s_n^{(m+1)} r^{-1} - s^{(m)}}{s_n^{(m+1)} - s_1^{(m+1)}} c_1^{(m+1)} + \frac{s^{(m)} - s_1^{(m+1)} r^{-1}}{s_n^{(m+1)} - s_1^{(m+1)}} c_n^{(m+1)} \leq \frac{s_n^{(m+1)} r^{-1} - s^{(m)}}{s_n^{(m+1)} - s_1^{(m+1)}} \overline{c}_1^{(m+1)} + \frac{s^{(m)} - s_1^{(m+1)} r^{-1}}{s_n^{(m+1)} - s_1^{(m+1)}} \overline{c}_n^{(m+1)}$ and $\frac{s_{j^*+1}^{(m+1)} r^{-1} - s^{(m)}}{s_{j^*+1}^{(m+1)} - s_{j^*}^{(m+1)}} c_{j^*}^{(m+1)} + \frac{s^{(m)} - s_{j^*}^{(m+1)} r^{-1}}{s_{j^*+1}^{(m+1)} - s_{j^*}^{(m+1)}} c_{j^*+1}^{(m+1)} \geq \frac{s_{j^*+1}^{(m+1)} r^{-1} - s^{(m)}}{s_{j^*+1}^{(m+1)} - s_{j^*}^{(m+1)}} \underline{c}_{j^*}^{(m+1)} + \frac{s^{(m)} - s_{j^*}^{(m+1)} r^{-1}}{s_{j^*+1}^{(m+1)} - s_{j^*}^{(m+1)}} \underline{c}_{j^*+1}^{(m+1)}$. So let

$$\overline{c}^{(m)} = \frac{s_n^{(m+1)} r^{-1} - s^{(m)}}{s_n^{(m+1)} - s_1^{(m+1)}} \overline{c}_1^{(m+1)} + \frac{s^{(m)} - s_1^{(m+1)} r^{-1}}{s_n^{(m+1)} - s_1^{(m+1)}} \overline{c}_n^{(m+1)}$$

and let

$$\underline{c}^{(m)} = \frac{s_{j^*+1}^{(m+1)} r^{-1} - s^{(m)}}{s_{j^*+1}^{(m+1)} - s_{j^*}^{(m+1)}} \underline{c}_{j^*}^{(m+1)} + \frac{s^{(m)} - s_{j^*}^{(m+1)} r^{-1}}{s_{j^*+1}^{(m+1)} - s_{j^*}^{(m+1)}} \underline{c}_{j^*+1}^{(m+1)}.$$

Continuing inductively in this way, one arrives at

$$\overline{c}^{(0)} = r^{-M} \sum_{j=1}^{M} \binom{M}{j} \theta^j (1-\theta)^{M-j} \max\left[0, (\frac{s_n^{(1)}}{s^{(0)}})^j (\frac{s_1^{(1)}}{s^{(0)}})^{N-j} s - x\right],$$

$$\underline{c}^{(0)} = r^{-M} \sum_{j=1}^{M} \binom{M}{j} \alpha^j (1-\alpha)^{M-j} \max\left[0, (\frac{s_{j^*+1}^{(1)}}{s^{(0)}})^j (\frac{s_{j^*}^{(1)}}{s^{(0)}})^{N-j} s - x\right],$$

where $\theta = \frac{s^{(0)} - s_1^{(1)} r^{-1}}{s_n^{(1)} - s_1^{(1)}}$ and $\alpha = \frac{s^{(0)} - s_{j^*}^{(1)} r^{-1}}{s_{j^*+1}^{(1)} - s_{j^*}^{(1)}}$.

3.3.4 Stochastic Dominance Option Bounds

Ritchken[28] derives bounds on the price of a call when the representative investor has a concave utility function. As explained in section 2.2, the current price of a given underlying asset S_0 can be described by the time discounted expectation $S_0 = r^{-1}E[DS]$, where S is the price of the asset after one period, r is the riskless discount factor, and $D = D(S)$ is the stochastic discount factor measurable S. Under the multinomial lattice, D and S have finite range. It is also assumed consumption increases in the stock price (in other words, the stock has positive correlation with the general level of wealth in the economy). Moreover, under the consumption-based model[31], $D = \frac{u'(K)}{\mathbb{E}[u'(K)]}$, where K is consumption. A security is said to be a positive beta security if $\mathbb{E}[K|S]$ is non-decreasing in S. Since u is concave and non-decreasing, D is positive and non-increasing in consumption. Therefore, $\mathbb{E}[D|S]$ is non-increasing in S if S describes the payoff of a positive beta security. Therefore, for any option on S with payoff $f(S) =: C$, the current price C_0 of the option is $C_0 = r^{-1}\mathbb{E}[Df(S)] = r^{-1}\mathbb{E}[\mathbb{E}[D|S]f(S)]$, where $\mathbb{E}[D|S]$ is positive non-increasing in S, $r^{-1}\mathbb{E}[\mathbb{E}[D|S]S] = r^{-1}\mathbb{E}[DS] = S_0$, and $\mathbb{E}[D] = \frac{\mathbb{E}[u'(K)]}{\mathbb{E}[u'(K)]} = 1$. Therefore, after one period in discrete probability space, the upper bound and lower bound for the price of a call on a positive-beta security is given by solving the following two linear programming problems:

$$\max_{\{d_i\}} c \qquad \min_{\{d_i\}} c,$$

with constraints

$$c = \sum_i c_i p_i d_i$$

$$\sum_i p_i d_i = r^{-1},$$

$$\sum_i s_i p_i d_i = s,$$

$$d_1 \geq d_2 \geq ... \geq d_n \geq 0,$$

where p_i is the probability of state i occurring, d_i is the discount rate if state i occurs, and $s_{i+1} > s_i, \forall i$. Set $w_i = d_i - d_{i+1}, i = 1, ..., n-1$ and

$w_n = d_n$. Then the problem can be restated:

$$\max_{\{w_i\}}(\min_{\{w_i\}})c = \sum_{j=1}^{n}(\sum_{i=1}^{j} c_i p_i)w_j,$$

with constraints

$$\sum_{j=0}^{n}(\sum_{i=1}^{j} p_i)w_j = r^{-1},$$

$$\sum_{j=0}^{n}(\sum_{i=1}^{j} s_i p_i)w_j = s,$$

$$w_j \geq 0, \qquad \forall j.$$

Now let $v_i = (\sum_{j=1}^{i} p_j)w_j$. Then the problem can be restated again:

$$\max_{\{v_i\}}(\min_{\{v_i\}})c = \sum_{j=1}^{n} \hat{c}_j v_j,$$

with constraints

$$\sum_{j=0}^{n} v_j = r^{-1},$$

$$\sum_{j=0}^{n} \hat{s}_j w_j = s,$$

$$w_j \geq 0, \qquad \forall j,$$

where $\hat{c}_j = \frac{\sum_{i=1}^{j} p_i c_i}{\sum_{i=1}^{j} p_i}$ and $\hat{s}_j = \frac{\sum_{i=1}^{j} p_i s_i}{\sum_{i=1}^{j} p_i}$. But this resembles the no-arbitrage linear programming problem. Based on the solutions to the no-arbitrage problem, one arrives at the following result find for the lower bound $\overline{c}$ and upper bound $\overline{c}$:

$$\overline{c} = r^{-1}(\hat{\theta}\hat{c}_n + (1-\hat{\theta})\hat{c}_1),$$

$$\underline{c} = r^{-1}(\hat{\alpha}\hat{c}_{h+1} + (1-\hat{\alpha})\hat{c}_h),$$

where $\hat{c}_j = \frac{\sum_{i \leq j} p_i c_i}{\sum_{i \leq j} p_i}$, $\hat{\theta} = \frac{sr - \hat{s}_1}{\hat{s}_n - \hat{s}_1}$, $\hat{\alpha} = \frac{sr - \hat{s}_h}{\hat{s}_{h+1} - \hat{s}_h}$, $h = \max\{j : \hat{s}_j < sr\}$.

3.3.5 Continuous Space Stochastic Dominance Bounds

Let the random variable C represent the payoff of a call at expiry. Let the continuous random variable S represent the stock price at expiry, let $\bar{s}$ be some constant such that $\mathbb{E}[S|S<\bar{s}] = sr$. To obtain the bounds, one can do something similar to what was done in section 3.3.3 for the no-arbitrage bounds, where the limit over discrete space was taken. Fix $\alpha > 0$ and consider a set of partitions $\{\mathcal{P}_\Delta\}_{\Delta\in(0,\alpha]}$ such that $\mathcal{P}_\Delta = \{s_1, ..., s_{h(\Delta)}, s_{h(\Delta)+1}, ..., s_{n(\Delta)}\}$ discretizes the range of S with a corresponding set of probabilites $\{p_1, ..., p_{n(\Delta)}\}$ that discretizes the distribution of S over the first lattice. Furthermore, construct a lattice $\{c_1, ..., c_{n(\Delta)}\}$ such that $c_j = (s_j - x)^+$, where x is the strike price, for all j. Let $\hat{c}_j$ and $\hat{s}_j$ be defined as in the previous section for all j and let $\Delta := \max_{j\in\{1,...,n-1\}} |s_{j+1} - s_j|$. Bounds can be found for the continuous case by taking the limit of the bounds as $\Delta \to 0$. For any $N \in \mathbb{N}$, there exists $\epsilon > 0$ such that $\Delta < \epsilon$ implies $0 \le r - \frac{\hat{s}_h}{s} < 2^{-N}$, $0 < \frac{\hat{s}_{h+1}}{s} - r < 2^{-N}$, $0 \le \hat{c}_{h+1} - \mathbb{E}[C|S<\bar{s}] < 2^{-N}$ $0 \le \mathbb{E}[C|S<\bar{s}] - c_h < 2^{-N}$, where the last two inequalities hold because the call price is continuous in the stock price. Now define constants $\delta, \delta', \epsilon', \epsilon'' > 0$ such that $r - \frac{\hat{s}_h}{s} =: \delta$, $\frac{\hat{s}_{h+1}}{s} - r =: \delta'$, $\hat{c}_{h+1} - \mathbb{E}[C|S<\bar{s}] =: \epsilon'$, and $\mathbb{E}[C|S<\bar{s}] - c_h =: \epsilon''$. Then, by the previous section, the lower bound price is

$$\begin{aligned}\underline{c} =& \frac{r^{-1}s(r+\delta') - s}{s(1+\delta') - s(1-\delta)}(\mathbb{E}[C|S<\bar{s}] + \epsilon') + \frac{s - r^{-1}s(r-\delta)}{s(1+\delta') - s(1-\delta)}(\mathbb{E}[C|S<\bar{s}] - \epsilon'') \\ =& r^{-1}\mathbb{E}[C|S<\bar{s}] + \frac{r^{-1}s(\delta'\epsilon' + \delta\epsilon'')}{\delta' + \delta}.\end{aligned}$$

But $0 \le \frac{r^{-1}s(\delta'\epsilon'+\delta\epsilon')}{\delta'+\delta} \le r^{-1}s2^{-N+1}$. So $r^{-1}\mathbb{E}[C|S<\bar{s}] \le \underline{c} \le r^{-1}\mathbb{E}[C|S<\bar{s}] + r^{-1}s2^{-N+1}$. Since N can be arbitrarily large, this yields $\lim_{\Delta\to 0}\underline{c} = r^{-1}\mathbb{E}[C|S<\bar{s}]$.

For the upper bound, fix $s_1 = \inf S$. Then for any $N \in \mathbb{N}$, there exists $\epsilon > 0$ such that $\Delta < \epsilon$ implies there exists $-2^{-N} \le \delta < 2^{-N}$ and $-2^{-N} \le \delta' < 2^{-N}$ such that $\mathbb{E}[\frac{S}{s}] - \frac{\hat{s}_n}{s} = \delta$ and $\mathbb{E}[\frac{C}{s}] - \frac{\hat{c}_n}{s} = \delta'$ where the second inequality holds because the call price is continuous in the stock price. Then the upper bound price is

$$\bar{c} = \frac{s - r^{-1}\inf S}{\mathbb{E}[S] - s\delta - \inf S}\mathbb{E}[C] + \frac{r^{-1}\mathbb{E}[S] - s}{\mathbb{E}[S] - s\delta - \inf S}\inf C.$$

So

$$\bar{c} \in \left[\frac{s - r^{-1}\inf S}{\mathbb{E}[S] - s2^{-N} - \inf S}\mathbb{E}[C] + \frac{r^{-1}\mathbb{E}[S] - s}{\mathbb{E}[S] - s2^{-N} - \inf S}\inf C, \frac{s - r^{-1}\inf S}{\mathbb{E}[S] + s2^{-N} - \inf S}\mathbb{E}[C] + \frac{r^{-1}\mathbb{E}[S] - s}{\mathbb{E}[S] + s2^{-N} - \inf S}\inf C\right]$$

$$\cup \left[\frac{s - r^{-1}\inf S}{\mathbb{E}[S] + s2^{-N} - \inf S}\mathbb{E}[C] + \frac{r^{-1}\mathbb{E}[S] - s}{\mathbb{E}[S] + s2^{-N} - \inf S}\inf C, \frac{s - r^{-1}\inf S}{\mathbb{E}[S] - s2^{-N} - \inf S}\mathbb{E}[C] + \frac{r^{-1}\mathbb{E}[S] - s}{\mathbb{E}[S] - s2^{-N} - \inf S}\inf C\right].$$

Since N can be arbitrarily large, $\lim_{\Delta\to 0} \bar{c} = \frac{s - r^{-1}\inf S}{\mathbb{E}[S] - \inf S}\mathbb{E}[C] + \frac{r^{-1}\mathbb{E}[S] - s}{\mathbb{E}[S] - \inf S}\inf C$. Observe that the risk-neutral CDFs $\underline{G}, \overline{G}$ implied by the lower and upper bounds, respectively, are

$$\underline{G}(x) = \frac{\min[F(\bar{s}), F(x)]}{F(\bar{s})},$$

$$\overline{G}(x) = \theta F(x) + (r^{-1} - \theta)\chi_{x > \inf S},$$

where $\theta = \frac{s - r^{-1}\inf S}{\mathbb{E}[S] - \inf S}$ and F is the physical CDF. Furthermore, the lower $\underline{D}$ and upper $\overline{D}$ bound time-discounted pricing kernels are given by

$$\underline{D} = \frac{\chi_{S < \bar{s}}}{r\mathbb{P}(S < \bar{s})}$$

$$\overline{D} = \theta + \frac{r^{-1} - \theta}{f(\inf S)}\delta_{\inf S},$$

where δ_x is the dirac-delta function centered at x and f is the physical PDF.

3.3.6 Multiple Revision Opportunities

When there are multiple portfolio revision opportunities before the expiry date, the price process of the underlying asset $S = (S_i : i = 0, 1, ..., m)$ can be modeled as a multiple-period discrete-time geometric process with stationary and independent increments adapted to the filtration $(\mathcal{F}_i : i = 1, ..., m)$. Define a stochastic process $C = (C_i : i = 0, 1, ..., m)$ such that

$C_i = C_i(S_i)$ is measurable in S_i. To find bounds on C_0 under a multi-period process, one can iteratively solve the linear programming problem in section 3.3.4, starting from $i = m$ and returning to $i = 0$. Therefore one obtains the following inequalities from the upper and lower bounds:

$$C_0 \le \mathbb{E}[\overline{D}_1 C_1(S_1)] = \mathbb{E}[\overline{D}_1 \mathbb{E}[\overline{D}_2 C_2(S_2)|\mathcal{F}_1]] = \mathbb{E}[\overline{D}_1 \mathbb{E}[\overline{D}_2 .... \mathbb{E}[\overline{D}_m C_m(S_m)|\mathcal{F}_m] ... \mathcal{F}_1]] = \mathbb{E}[\overline{D}_1 \overline{D}_2 .... \overline{D}_m C_m(S_m)] = \mathbb{E}[\overline{D} C_m(S_m)]$$

$$C_0 \ge \mathbb{E}[\underline{D}_1 C_1(S_1)] = \mathbb{E}[\underline{D}_1 \mathbb{E}[\underline{D}_2 C_2(S_2)|\mathcal{F}_1]] = \mathbb{E}[\underline{D}_1 \mathbb{E}[\underline{D}_2 .... \mathbb{E}[\underline{D}_m C_m(S_m)|\mathcal{F}_m] ... \mathcal{F}_1]] = \mathbb{E}[\underline{D}_1 \underline{D}_2 .... \underline{D}_m C_m(S_m)] = \mathbb{E}[\underline{D} C_m(S_m)],$$

where $\overline{D} := \prod_{i=1}^m \overline{D}_i$ and $\underline{D} := \prod_{i=1}^m \underline{D}_i$, where $\overline{D}_i$ and $\underline{D}_i$ are $\mathcal{F}_i$ measurable for $i = 1, ..., m$. They are also iid since S is (geometrically) stationary with independent increments. Therefore, stochastic dominance bounds with multiple revision opportunities may be obtained iteratively as well.

3.4 Risk-Neutral Processes

Let the constant T represent the expiry time of a call and let $\Delta t > 0$ represent some time increment. Define a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\mathbf{F} := \{\mathcal{F}_t\}_{t \in [0,T]}$. $\mathbb{E}_t[\cdot]$ will represent the expectation operator conditional on the filtration at time t. Unless stated otherwise, all stochastic processes will be adapted to the filtration $\mathbf{F}$. In the following section, it will be shown that certain discrete-time processes converge to certain infinitesimal-time processes in the limit as the change in time goes to 0. Let $s^{(n)}_{t^{(n)}_k}$ represent a sequence of discrete-time processes indexed by n at time $t^{(n)}_k$, where $k \in \{0, ..., m_n\}$, where $0 = t^{(n)}_0 < t^{(n)}_1 < ... < t^{(n)}_{m_n} = T$ and $\max_{k \in \{1,...,m_n\}} |t^{(n)}_k - t^{(n)}_{k-1}| \to 0$ as $n \to \infty$. Observe that convergence in distribution is equivalent to weak convergence. So, for any $r \in [0, T]$, let h be some index such that $t^{(n)}_h = \min\{t \in \{t^{(n)}_0, ..., t^{(n)}_{m_n}\} : t \ge r\}$ for any n. Then

$$s_0 + \lim_{n\to\infty} \sum_{k=1}^{h} (s^{(n)}_{t^{(n)}_k} - s^{(n)}_{t^{(n)}_{k-1}}) \overset{d}{=} s_0 + \int_0^r ds_t \Leftrightarrow \lim_{n\to\infty} \mathbb{E}[f(s_0 + \sum_{k=1}^{h} (s^{(n)}_{t_k} - s^{(n)}_{t_{k-1}}))] = \mathbb{E}[f(s_0 + \int_0^r ds_t)],$$

for any continuous bounded function f. Now, for any fixed n, define $\{t^{(n)}_0, ..., t^{(n)}_{m_n}\}$ so that $\frac{T}{m_n} = t_k - t_{k-1} =: \Delta t_n$ for all $k \in \{1, ..., m_n\}$. Since the number of terms in the sum is increasing linearly in $\frac{1}{\Delta t_n}$, each term in the sum must converge faster than linearly in Δt_n. In other words, the discrete-time process converges to the corresponding continuous-time process if and only if, for every k,

$$\mathbb{E}[\Delta f_k^{(n)} - \int_{t_{k-1}}^{t_{k-1}+\Delta t_n} df] = o(\Delta t_n),$$

where $\Delta f_n^{(n)} = f(s_0 + \sum_{j=1}^{k}(s_{t_j^{(n)}}^{(n)} - s_{t_{j-1}}^{(n)})) - f(s_0 + \sum_{j=1}^{k-1}(s_{t_j^{(n)}}^{(n)} - s_{t_{j-1}^{(n)}}^{(n)}))$ and $df(s_t) = f(s_t + ds_t) - f(s_t)$. Therefore, the discrete-time process converges to the corresponding continuous-time process if and only if

$$\lim_{n\to\infty} \mathbb{E}[\frac{\Delta f_{t_k^{(n)}}^{(n)}}{\Delta t_n}] = \mathbb{E}[\frac{df(s_t)}{dt}],$$

where $t_k^{(n)} = \max\{r \in \{t_0^{(n)}, ..., t_{m_n}^{(n)}\} : r \leq t\}$. Now let $N : \mathbb{R}^+ \to \mathbb{N}$ be some non-increasing function such that $N(\Delta t) = n$ when $\Delta t_n = \Delta t$. Then the discrete time process converges in distribution to the continuous time process if and only if

$$\lim_{\Delta t\to 0} \mathbb{E}[\frac{\Delta f_{t_k^{(n)}}^{(N(\Delta t))}}{\Delta t}] = \mathbb{E}[\frac{df(s_t)}{dt}].$$

Now, fix $\Delta t = \Delta t_n$ and fix $t \in \{t_0^{(n)}, ..., t_{m_n}^{(n)}\}$. Define the operator $\mathcal{A}_{\Delta t}^{s_t}$ such that

$$\mathcal{A}^{s_t} f := \lim_{n\to\infty} \frac{\mathbb{E}_t[f(s_{t+\Delta t}^{(n)})] - f(s_t^{(n)})}{\Delta t}.$$

Then, since weak convergence is equivalent to convergence in distribution it must hold that for any closed and bounded f, $\mathcal{A}^{s_t} f = \mathbb{E}_t[\frac{df}{dt}]$ if and only if $s_0 + \sum_{k=1}^{h}(s_{t_k^{(N(\Delta t))}}^{(N(\Delta t))} - s_{t_{k-1}^{(N(\Delta t))}}^{(N(\Delta t))}) \xrightarrow{d} s_0 + \int_0^r ds_t$ as $\Delta t \to 0$. For future reference, this is the kind of convergence being referred to when it is said that $s_{t_k^{(n)}}^{(n)} - s_{t_{k-1}^{(n)}}^{(n)}$ "converges to" some infinitesimal ds_t or when it is said that ds_t is "the limit" of $s_{t_k^{(n)}}^{(n)} - s_{t_{k-1}^{(n)}}^{(n)}$ as $\Delta t \to 0$. Furthermore, for notational simplicity, from now on, fix some value t in $[0, T)$ and some small $\Delta t > 0$, and let $s_{t+\Delta t} := s_{t_k^{(N(\Delta t))}}^{(N(\Delta t))}$ and $s_t := s_{t_{k-1}^{(N(\Delta t))}}^{(N(\Delta t))}$, where $t_{k-1}^{(N(\Delta t))} = \max\{r \in \{t_0^{(n)}, ..., t_{m_{N(\Delta t)}}^{(n)}\} : r \leq t\}$.
Now, the Stone-Weierstrass Theorem states that the space of smooth functions on $\mathbb{R}$ is dense on the space of continuous bounded functions on $\mathbb{R}$ under the uniform norm. So, for any continuous bounded f with support on $\mathbb{R}$, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of smooth functions that converges uniformly on any compact set, and therefore pointwise on $\mathbb{R}$, to f. So, if

$\mathcal{A}^{s_t} f = \mathbb{E}_t[\frac{df}{dt}]$ for any smooth f then, for any continuous and bounded f,

$$\mathcal{A}^{s_t} f = \mathcal{A}^{s_t} \lim_{n\to\infty} f_n = \lim_{n\to\infty} \mathcal{A}^{s_t} f_n = \lim_{n\to\infty} \mathbb{E}_t[\frac{df_n}{dt}] = \lim_{n\to\infty} \mathbb{E}_t[\frac{df_n}{dt}] = \mathbb{E}_t[\lim_{n\to\infty} \frac{df_n}{dt}] = \mathbb{E}_t[\frac{df}{dt}].$$

Therefore, $\mathcal{A}^{s_t} f = \mathbb{E}_t[\frac{df}{dt}]$ for smooth f is a sufficient condition for convergence in distribution to the corresponding infinitesimal.
Observe, from Ito's lemma for a geometric Weiner processes with drift, if f is smooth, then

$$\mathbb{E}_t[\frac{df}{dt}] = \mu f'(s_t)s_t + \sigma^2 s_t^2 f''(s_t),$$

where μ and σ^2 are the drift and variance, respectively, of the Weiner process. According to Merton[24], Δs_t satisfies the Lindeberg condition if, for all $\delta > 0$, $\frac{1}{\Delta t}\int_{|x|>\delta} d\mathbb{P}[\Delta s_t = x] \to 0$ as $\Delta t \to 0$. This is a sufficient condition for $\frac{\Delta s_t}{\Delta t}$ to converge to a Gaussian random variable as $\Delta t \to 0$. If Δs_t satisfies the Lindeberg condition for all t in the simplex then

$$\begin{aligned}
\mathcal{A}^{s_t} f &= \lim_{\Delta t\to 0}(\frac{f'(s_t)\mathbb{E}_t[\Delta s_t]}{\Delta t} + \frac{1}{2}\frac{f''(s_t)\mathbb{E}_t[\Delta s_t^2]}{\Delta t} + O(\frac{\Delta s_t^3}{\Delta t}))\\
&= \lim_{\Delta t\to 0}(f'(s_t)\mathbb{E}_t[\frac{\Delta s_t}{\Delta t}] + \frac{1}{2}\frac{f''(s_t)\text{Var}([\Delta s_t])}{\Delta t} + \frac{1}{2}\frac{f''(s_t)\mathbb{E}_t[\Delta s_t]^2}{\Delta t} + O(\mathbb{E}_t[\frac{\Delta s_t^3}{\Delta t}]))\\
&= f'(s_t)\mu + \frac{1}{2}\sigma^2 f''(s_t),
\end{aligned}$$

where $\mu = \lim_{\Delta t\to 0} \mathbb{E}_t[\frac{\Delta s_t}{\Delta t}]$, $\sigma^2 = \lim_{\Delta t\to 0} \frac{\text{Var}(\Delta s_t)}{\Delta t}$, and the other terms cancel because the moments of the normal distribution require $\mathbb{E}_t[\Delta s_t]^2 = O(\Delta t^2)$ and $O(\mathbb{E}_t[\frac{\Delta s_t^3}{\Delta t}]) = O(\Delta t^{1/2})$. And so Δs converges to a diffusive SDE. For a Jump-Diffusion process of the form

$$ds_t = \mu s_t dt + \sigma s_t dw_t + J_t s_t dN_t,$$

where J is an independent jump size and dN_t is an independent Poisson process differential, it follows from Ito's Lemma for Jump-Diffusive processes that, for any function f,

$$\mathbb{E}_t[\frac{df}{dt}] = \mu f'(s_t)s_t + \sigma^2 s_t^2 f''(s_t) + \lambda\mathbb{E}_t[f((1+J_t)s_t) - f(s_t)].$$

Given some $\mathcal{F}_t$-measurable pricing kernel $D_{t,t+\Delta t}$, define a function γ_t : $\mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ such that $\gamma_t(x, \cdot) =: \gamma_t(x)$ is $\mathcal{F}_t$-measurable for any $x \in \mathbb{R}$ and $\gamma_t(x) = \{D_{t,t+\Delta t} | s_{t+\Delta t} = x\}$. Then

$$\mathbb{E}_t[D_{t,t+\Delta t} f(s_{t+\Delta t})] = \int_0^\infty \gamma_t(x) f(x) p_t(x) dx = R^{-\Delta t} \int_0^\infty f(x) q_t(x) dx,$$

where p_t is the $\mathcal{F}_t$-measurable physical density of $s_{t+\Delta t}$ conditioning on s_t and $q_t := R^{\Delta t} \gamma_t p_t$ is also a density since

$$R^{-\Delta t} = \mathbb{E}[D_{t,t+\Delta t}] = \int_0^\infty \gamma(x) p(x) dx = R^{-\Delta t} \int_0^\infty q(x) dx.$$

Rearranging yields $\int_0^\infty q(x)dx = 1$. Also, it follows that $q > 0$ because $\gamma > 0$. So define a new random variable $s^q_{t+\Delta t}$ that is measurable in $s_{t+\Delta t}$ and has density q. Define the risk-neutral process differential ds^q_t as the limit of $s^q_{t+\Delta t} - s_t$ as $\Delta t \to 0$. The goal of the following sections will be to find the risk-neutral process differentials generated by the upper and lower bound pricing kernels given a specified physical process differential.

3.4.1 Diffusive Risk-Neutral Bound Process

Let

$$s_{t+\Delta t} = s_t + \mu(s_t, t)\Delta t + \sigma(s_t, t) B \sqrt{\Delta t},$$

where

$$B = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

is independent of s_t and $\mu, \sigma > 0$ are deterministic functions. Let $p_{\Delta t}(x, \cdot)$ be the PDF of $s_{t+\Delta t}$ given $s_t = x$. It can be observed that, for any $\delta > 0$, $s_{t+\Delta t} - s_t \leq \mu(s_t, t)\Delta t + \sigma(s_t, t)\sqrt{\Delta t} < \delta$ almost surely when $\Delta t \in (0, \frac{(-\sigma + \sqrt{\sigma^2 + 4\mu\delta})^2}{4\mu^2})$. Since $\frac{(-\sigma+\sqrt{\sigma^2+d\mu\delta})^2}{4\mu^2} > 0$, the interval is non-empty. Therefore, $\int_{|y-x|>\delta} p_{\Delta t}(x, y) dy = 0$ for positive Δt small enough. So the process satisfies the Lindeberg condition and converges to a diffusive process as $\Delta t \to 0$. Observe that $\mathbb{E}_t[s_{t+\Delta t} - s_t] = \mu(s_t, t)\Delta t$ and $\text{Var}_t(s_{t+\Delta t} - s_t) = \sigma^2(s_t, t)\Delta t$. So the discrete time-process converges to a continuous-time process with the following differential:

$$ds_t = \mu(s_t, t)dt + \sigma(s_t, t)dw_t,$$

where w_t is a Weiner process at time t. Now let s_t^u be the risk-neutral process at time t. Observe that the upper bound pricing kernel $m_{t,t+\Delta t}$ at time t for a random variable measurable at time $t+\Delta t$ is described by $m_{t,t+\Delta t} = \theta(s_t, t) + \frac{R^{-\Delta t} - \theta(s_t,t)}{p(s_t, \inf_t s_{t+\Delta t})}\delta_{\inf s_{t+\Delta t}}$, where R is the riskless discount factor, $\theta = \frac{s_t - R^{-\Delta t}\inf_t s_{t+\Delta t}}{\mathbb{E}_t[s_{t+\Delta t}] - \inf_t s_{t+\Delta t}}$, and δ_x is the dirac-delta function at x. Observe that if $s^u_{t+\Delta t} = R^{\Delta t} m_{t,t+\Delta t} s_{t+\Delta t}$. Then

$$\begin{aligned} s^u_{t+\Delta t} =& s_t + \mu(s_t,t)\Delta t + R^{\Delta t}\theta(s_t,t)\sigma(s_t,t)B\sqrt{\Delta t} - (1 - R^{\Delta t}\theta(s_t,t))\sigma(s_t,t)\sqrt{\Delta t} \\ \stackrel{d}{=}& s_t + \mu(s_t,t)\Delta t + \sigma(s_t,t)B^u\sqrt{\Delta t}, \end{aligned}$$

where

$$B^u = \begin{cases} 2R^{\Delta t}\theta(s_t,t) - 1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

and

$$\begin{aligned} \theta(s_t,t) =& \frac{s_t - R^{-\Delta t}(s_t + \mu(s_t,t)\Delta t - \sigma(s_t,t)\sqrt{\Delta t})}{s_t + \mu(s_t,t)\Delta t - (s_t + \mu(s_t,t)\Delta t - \sigma(s_t,t)\sqrt{\Delta t})} \\ =& \frac{(1 - R^{-\Delta t}) - \frac{R^{-\Delta t}\mu(s_t,t)}{s_t}\Delta t - \frac{R^{-\Delta t}\sigma(s_t,t)}{s_t}\sqrt{\Delta t}}{-\frac{\sigma(s_t,t)}{s_t}\sqrt{\Delta t}} \\ =& \frac{(R^{-\Delta t} - 1) + \frac{R^{-\Delta t}\mu(s_t,t)}{s_t}\Delta t}{\frac{\sigma(s_t,t)}{s_t}\sqrt{\Delta t}} + R^{-\Delta t}. \end{aligned}$$

Observe that $\theta \to 0$ as $\delta \to 0$, so B first-order dominates B^u for Δt small enough. Moreover, by symmetry, $-B$ also first-order dominates B^u for Δt small enough. Therefore, if $q_{\Delta t}(x, \cdot)$ is the risk-neutral density of $s^u_{t+\Delta t}$ given $s_t = x$ then, for all $\delta > 0$, $\int_{|y-x|>\delta} q_\Delta(x,y)dy \leq \int_{|y-x|>\delta} p_\Delta(x,y)dy$ for Δt small enough. Therefore, $s^u_{t+\Delta t}$ also satisfies the Lindeberg condition. Observe that $\mathbb{E}_t[s^u_{t+\Delta t} - s_t] = R^{\Delta t} - 1 = rs_t\Delta t + O(\Delta t^2)$, where $r = \ln R$ is the instantaneous riskless rate, and $\text{Var}(s^u_{t+\Delta t} - s_t) = \sigma^2(s_t,t)\Delta t$. So the risk-neutral process is

$$ds_t^u = rs_t dt + \sigma(s_t, t)dw_t.$$

Now let

$$s_{t+\Delta t} = s_t + \mu(s_t, t)\Delta t + \sigma(s_t, t)Z\sqrt{\Delta t},$$

where Z is a standard normal random variable. Then $p_{t,t+\Delta}(x, y) = \frac{1}{\sqrt{2\pi\sigma(s_t,t)^2\Delta t}} e^{-\frac{(y-x-\mu(s_t,t)\Delta t)^2}{2\sigma(s_t,t)^2\Delta t}}$ and

$$\begin{aligned}
\lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{|y-x|>\delta} \frac{1}{\sqrt{2\pi\sigma(s_t, t)^2\Delta t}} e^{-\frac{(y-x-\mu(s_t,t)\Delta t)^2}{2\sigma(s_t,t)^2\Delta t}} dy =& \int_{\delta}^{\infty} \lim_{\Delta t\to 0} \frac{1}{\sqrt{2\pi\sigma(s_t, t)^2\Delta t^3}} e^{-\frac{(u-\mu(s_t,t)\Delta t)^2}{2\sigma(s_t,t)^2\Delta t}} du \\
&+ \int_{-\infty}^{\delta} \lim_{\Delta t\to 0} \frac{1}{\sqrt{2\pi\sigma(s_t, t)^2\Delta t^3}} e^{-\frac{(u-\mu(s_t,t)\Delta t)^2}{2\sigma(s_t,t)^2\Delta t}} du \\
=& \int_{\delta}^{\infty} \lim_{M\to\infty} \frac{M^{\frac{3}{2}}}{\sqrt{2\pi\sigma(s_t, t)}} e^{-\frac{(uM^2-\mu(s_t,t)M)^2}{2\sigma(s_t,t)^2}} du \\
&+ \int_{-\infty}^{\delta} \lim_{M\to\infty} \frac{M^{\frac{3}{2}}}{\sqrt{2\pi\sigma(s_t, t)}} e^{-\frac{(uM^2-\mu(s_t,t)M)^2}{2\sigma(s_t,t)^2}} du \\
=& \int_{\delta}^{\infty} \lim_{M\to\infty} \frac{M^{\frac{3}{2}}}{\sqrt{2\pi\sigma(s_t, t)}} e^{-\frac{u^2M^4}{2\sigma(s_t,t)^2}+O(M^3)} du \\
&+ \int_{-\infty}^{\delta} \lim_{M\to\infty} \frac{M^{\frac{3}{2}}}{\sqrt{2\pi\sigma(s_t, t)}} e^{-\frac{u^2M^4}{2\sigma(s_t,t)^2}+O(M^3)} du \\
=& 0,
\end{aligned}$$

where the limit can be taken inside the integral because the inside is integrable for any $\Delta t > 0$, as well as in the limit as Δt approaches 0. Therefore the discrete-time process satisfies the Lindeberg condition. Since the process has mean $s_t + \mu(s_t, t)\Delta t$ and variance $\sigma^2(s_t, t)\Delta t$, it converges to a process with the following differential:

$$ds = \mu(s_t, t)dt + \sigma(s_t, t)dw_t.$$

The lower bound pricing kernel $d_{t,t+\Delta t}$ at time t for a stock measurable at time $t + \Delta t$ is given by $d_{t,t+\Delta} = \frac{\chi_{s_{t+\Delta t}\leq s^*}}{R^{\Delta t}\mathbb{P}[s_{t+\Delta t}\leq s^*]}$, where s^* is chosen so that

$\mathbb{E}[R^{\Delta t} d_{t,t+\Delta} s_{t+\Delta t}] = R^{\Delta t} s_t$. If s^q_t is the risk-neutral process at time t then

$$\begin{aligned} s^q_{t+\Delta t} =& R^{\Delta t} d_{t,t+\Delta t} s_{t+\Delta t} \\ =& s_t + \mu(s_t,t)\Delta t + \sigma(s_t,t)\tilde{Z}\sqrt{\Delta t}, \end{aligned}$$

where $\tilde{Z} = \frac{Z\chi_{Z<\bar{z}}}{\mathbb{P}[Z<\bar{z}]}$, where $\bar{z}$ is chosen so that

$$s_t + \mu(s_t,t)\Delta t + \sigma(s_t,t)\mathbb{E}[\tilde{Z}]\sqrt{\Delta t} = R^{\Delta t} s_t.$$

Observe,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\bar{z}} z e^{-\frac{z^2}{2}} dz = \mathbb{E}[\tilde{Z}] = \frac{(R^{\Delta t}-1)s_t - \mu(s_t,t)\Delta t}{\sigma(s_t,t)\sqrt{\Delta t}} = (\frac{rs_t - \mu(s_t,t)}{\sigma(s_t,t)})\sqrt{\Delta t} + O(\Delta t),$$

which goes to $0 = \mathbb{E}[Z]$ as $\Delta t \to 0$. Moreover,

$$\bar{z} \geq \frac{\mathbb{E}[Z\chi_{Z\leq\bar{z}}]}{\mathbb{P}[Z \leq \bar{z}]} = \mathbb{E}[\tilde{Z}] = (\frac{rs_t - \mu(s_t,t)}{\sigma(s_t,t)})\sqrt{\Delta t} + O(\Delta t).$$

These two facts imply that $\bar{z} \to \infty$ as $\Delta t \to 0$ and so

$$\mathrm{Var}(\tilde{Z}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\bar{z}} z^2 e^{-\frac{z^2}{2}} dz - (\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\bar{z}} z e^{-\frac{z^2}{2}} dz)^2 \overset{\Delta t \to 0}{\to} 1,$$

which implies $\mathrm{Var}(\tilde{Z}) = 1 + o(1)$. Therefore,

$$\begin{aligned} \mathrm{Var}(s_{t+\Delta t}) =& \sigma(s_t,t)^2 \mathrm{Var}(\tilde{Z})\Delta t \\ =& \sigma(s_t,t)^2 \Delta t(1 + o(1)) \\ =& \sigma(s_t,t)^2 \Delta t + o(\Delta t). \end{aligned}$$

Therefore, the lower-bound risk-neutral process is

$$ds^q_t = rs_t dt + \sigma(s_t,t)dw_t,$$

which agrees with the upper bound process.

3.4.2 Risk-Neutral Jump Diffusion Processes

Let

$$s_{t+\Delta t} = s_t + (\mu(s_t,t) - k(s_t,t)\lambda)\Delta t(1-\Delta N) + \sigma(s_t,t)B\sqrt{\Delta t}(1-\Delta N) + J\Delta N,$$

where B is defined as in the previous section, $\Delta N \sim \text{binom}(\lambda\Delta t)$ and is independent of B and J is a continuous with mean $k(s_t,t)$ and is independent of B and ΔN. Observe that if $p(x,\cdot)$ is the PDF of $s_{t+\Delta t}$ given that $s_t = x$, $p_D(x,\cdot)$ is the PDF of $s_{t+\Delta t}$ given $\Delta N = 0$ and $s_t = x$, and p_J is the PDF of J then

$$\frac{1}{\Delta t}\int_{|y-x|>\delta} p(x,y)dy = (\frac{1}{\Delta t} - \lambda)\int_{|y-x|>\delta} p_D(x,y)dy + \lambda\int_{|y|>\delta} p_J(y)dy.$$

As demonstrated in the previous section, the first term in the above sum goes to 0 as Δt goes to 0. However, since J is independent of Δt, $\int_{|y|>\delta} p_J(y)dy > 0$ for some $\delta \in (0, \min[|\sup J|, |\inf J|])$ as $\Delta t \to 0$ and so the Lindeberg condition is violated. However, it can be seen that

$$s_t + (\mu(s_t,t) - k(s_t,t)\lambda)\Delta t(1-\Delta N) + \sigma(s_t,t)Z\sqrt{\Delta t}(1-\Delta N)$$

still satisfies the Lindeberg condition and has mean $s_t + (\mu(s_t,t) - k(s_t,t)\lambda)\Delta t + O(\Delta t^{3/2})$ and variance $\sigma^2(s_t,t)\Delta t + O(\Delta t^{3/2})$. Moreover, observe that J is independent of Δt and $\mathbb{P}[\Delta N = 1]$ scales linearly with Δt. Now, define the following discrete-time processes:

$$s^J_{t+\Delta t} = s_t + J\Delta N$$

$$s^D_{t+\Delta t} = s_t + (\mu(s_t,t) - k(s_t,t)\lambda)\Delta t(1-\Delta N) + \sigma(s_t,t)^2 B\sqrt{\Delta t}(1-\Delta N)$$

Also, for any smooth f,

$$\begin{aligned}\mathcal{A}^{s_t^J} f = &\lim_{\Delta t \to 0^+} \frac{\mathbb{E}[f(s_t + J\Delta N) - f(s_t)]}{\Delta t} \\ = &\lim_{\Delta t \to 0^+} \lambda \Delta t \frac{\mathbb{E}[f(s_t + J) - f(s_t)]}{\Delta t} \\ = &\lambda \mathbb{E}[f(s_t + J) - f(s_t)]\end{aligned}$$

which, as discussed earlier, is the infinitesimal generator for a pure Jump process (i.e., a Jump-Diffusion process when $\sigma = \mu - k(s_t, t)\lambda = 0$) with Jump intensity λdt and Jump size J. Therefore,

$$s_{t+\Delta t}^J - s_t \to J dN =: ds_t^J.$$

Moreover, since the diffusive part satisfies the Lindeberg condition,

$$s_{t+\Delta t}^D - s_t \to (\mu(s_t, t) - k(s_t, t)\lambda)dt + \sigma(s_t, t)dw_t =: ds_t^D.$$

Moreover, observe that

$$s_{t+\Delta t} - s_t = (s_{t+\Delta t}^D - s_t) + (s_{t+\Delta t}^J - s_t) \to ds_t^D + ds_t^J = (\mu(s_t, t) - k(s_t, t)\lambda)dt + \sigma(s_t, t)dw_t + JdN,$$

which is a Jump-Diffusion process where $dN \sim \text{binom}(\lambda dt)$. Observe that the above process is a Jump-Diffusion process. Also observe that if Δt is small enough then $\inf s_{t+\Delta t} = s_t + (\mu(s_t, t) - k(s_t, t)\lambda)\Delta t + \inf J$ as long as $\inf J < 0$. So, for the upper bound risk-neutral process,

$$\begin{aligned} s_{t+\Delta t}^u = &\begin{cases} s_t + (\mu(s_t, t) - k(s_t, t)\lambda)\Delta t(1 - \Delta N) + \sigma(s_t, t)B\sqrt{\Delta t}(1 - \Delta N) + J\Delta N & \text{with probab} \\ s_t + \inf J & \text{with probab} \end{cases} \\ = &\begin{cases} s_t + (\mu(s_t, t) - k(s_t, t)\lambda)\Delta t + \sigma(s_t, t)B\sqrt{\Delta t} & \text{with probability } R^{\Delta t}\theta(1 - \lambda \Delta t) \\ s_t + J & \text{with probability } R^{\Delta t}\theta \lambda \Delta t \\ s_t + \inf J & \text{with probability } 1 - R^{\Delta t}\theta \end{cases} \\ = &s_t + (\mu(s_t, t) - k(s_t, t)\lambda)\Delta t(1 - \Delta \tilde{N}) + \sigma(s_t, t)B\sqrt{\Delta t}(1 - \Delta \tilde{N}) + \tilde{J}\Delta \tilde{N} \end{aligned}$$

where $\Delta\tilde{N} \sim \text{binom}(1 - R^{\Delta t}\theta + \lambda R^{\Delta t}\theta\Delta t)$ and

$$\tilde{J} = \begin{cases} J & \text{with probability } \frac{R^{\Delta t}\theta\lambda\Delta t}{1-R^{\Delta t}\theta+\lambda R^{\Delta t}\theta\Delta t} \\ \inf J & \text{with probability } \frac{1-R^{\Delta t}\theta}{1-R^{\Delta t}\theta+\lambda R^{\Delta t}\theta\Delta t} \end{cases},$$

where

$$\begin{aligned} R^{\Delta t}\theta =& R^{\Delta t}\theta(s_t, t) \\ =& R^{\Delta t}\left(\frac{s_t - R^{-\Delta t}(s_t + \mu(s_t,t)\Delta t + \inf J)}{s_t + \mu(s_t,t)\Delta t - (s_t + \mu(s_t,t)\Delta t + \inf J)}\right) \\ =& R^{\Delta t}\left(\frac{(1 - R^{-\Delta t}) - \frac{R^{-\Delta t}\mu(s_t,t)}{s_t}\Delta t - \frac{R^{-\Delta t}\inf J}{s_t}}{-\frac{\inf J}{s_t}}\right) \\ =& R^{\Delta t}\left(\frac{(R^{-\Delta t} - 1) + \frac{R^{-\Delta t}\mu(s_t,t)}{s_t}\Delta t}{\frac{\inf J}{s_t}} + R^{-\Delta t}\right) \\ =& \frac{(1 - R^{\Delta t})s_t + \mu(s_t,t)\Delta t}{\inf J} + 1 \\ =& 1 + \frac{\mu(s_t,t) - rs}{\inf J}\Delta t + O(\Delta t^2). \end{aligned}$$

Therefore, $\Delta\tilde{N} \sim \text{binom}(\frac{rs-\mu(s_t,t)+\lambda\inf J}{\inf J}\Delta t + O(\Delta t^2))$ and

$$\tilde{J} = \begin{cases} J & \text{with probability } \frac{\lambda\inf J}{\lambda\inf J+rs-\mu(s_t,t)} + O(\Delta t) \\ \inf J & \text{with probability } \frac{rs-\mu(s_t,t)}{\lambda\inf J+rs-\mu(s_t,t)} + O(\Delta t) \end{cases}.$$

Observe that the diffusive part is independent of the parameters of $\Delta\tilde{N}$ and so converges to the same Weiner process as before. Furthermore, observe that the infinitesimal generator of

$$s^{\tilde{J}}_{t+\Delta t} = s_t + \tilde{J}\Delta N$$

is given by

$$\begin{aligned}
\mathcal{A}^{s_t^{\tilde{J}}} f &= \lim_{\Delta t \to 0^+} \frac{\mathbb{E}[f(s_t + \tilde{J}\Delta N) - f(s_t)]}{\Delta t} \\
&= \lim_{\Delta t \to 0^+} \lambda^u \Delta t \frac{\mathbb{E}[f(s_t + \tilde{J}) - f(s_t)] + O(\Delta t)}{\Delta t} \\
&= \lambda^u \lim_{\Delta t \to 0^+} \mathbb{E}[f(s_t + \tilde{J}) - f(s_t)] + O(\Delta t) \\
&= \lambda^u \mathbb{E}[f(s_t + J^u) - f(s_t)]
\end{aligned}$$

where

$$J^u = \begin{cases} J & \text{with probability } \frac{\lambda \inf J}{\lambda \inf J + rs - \mu(s_t,t)} \\ \inf J & \text{with probability } \frac{rs - \mu(s_t,t)}{\lambda \inf J + rs - \mu(s_t,t)} \end{cases}$$

and $\lambda^u = \frac{rs - \mu(s_t,t) + \lambda \inf J}{\inf J}$, which is the infinitesimal generator for a process with jump intensity $\lambda^u dt$ and jump size J^u. Therefore, $s^u_{t+\Delta t} - s_t$ converges to

$$ds^u = (\mu(s_t,t) - k(s_t,t)\lambda)dt + \sigma(s_t,t)dw_t + J^u d\tilde{N},$$

where $d\tilde{N} \sim \text{binom}(\lambda^u dt)$. Observe that $\mathbb{E}[J^u d\tilde{N}] = (rs - \mu(s_t,t) + k(s_t,t)\lambda)dt$ and thus $\mathbb{E}[ds^u] = rsdt$, so the process is indeed risk-neutral. So, by Ito's Lemma for Jump-Diffusion Processes, if ds^u_t is the risk-neutral process differential, and $f : \mathbb{R}^2 \to \mathbb{R}$ is some smooth function then

$$\mathbb{E}_t[f(s_t + ds^u_t, t + dt) - f(s_t,t)] = (\frac{\partial f}{\partial t} + \mu(s_t,t)\frac{\partial f}{\partial x} + \sigma(s_t,t)^2 \frac{\partial^2 f}{\partial x^2} + \lambda^u \mathbb{E}_t[f(s_t + J^u) - f(s_t)])dt = rf(s_t,t)dt.$$

Rearranging, one arrives at the following PDE:

$$\frac{\partial f}{\partial t} + \mu(s_t,t)\frac{\partial f}{\partial s_t} + \sigma(s_t,t)^2 \frac{\partial^2 f}{\partial s_t^2} + \lambda^u \mathbb{E}_t[f(s_t + J^u) - f(s_t)] - rf = 0. \quad (3.3)$$

This generates the following Theorem:

Theorem 5 (Perrakis)**.** *Fix $t < T$, let s_t be represent the Jump-Diffusion process 3.4.2, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be some smooth function, convex in the first argument, such that $f(s_T, T) = g(s_T)$ and there exists a random variable D, measurable in ds_t, such that $\mathbb{E}_t[Dds_t] = 0, \mathbb{E}_t[D] = 1 - rdt, \mathbb{E}_t[Df(s_{t+dt}, t+dt)] = f(s_t,t)$, and D is decreasing in ds_t. If $\overline{f} : \mathbb{R}^2 \to \mathbb{R}$*

solves (3.3) with boundary condition $\overline{f}(s_T, T) = g(s_T)$ *then* $\overline{f}(s_t, t) \geq f(s_t, t)$.

Also observe that the PDE (3.3) is Merton's PDE when $\mu(s_t, t) = \mu s_t, \sigma(s_t, t) = \sigma s_t, \lambda = \lambda^u$, $k(s_t, t) = k s_t$, and $J^u = (J_t - 1)s_t$.
If Δt is small enough, the lower bound discrete-time risk-neutral process is given by

$$s^q_{t+\Delta t} = s_t + (\mu(s_t, t) - k\lambda)\Delta t(1 - \Delta N) + \sigma(s_t, t)B\sqrt{\Delta t}(1 - \Delta N) + \hat{J}\Delta N,$$

where $\hat{J} = \{J | J \leq \bar{j}\}$ where $\bar{j}$ are chosen so that

$$\mathbb{E}[s^d_{t+\Delta t}] = s_t + \mu(s_t, t)\Delta t(1 - \lambda\Delta t) + \mathbb{E}[\tilde{J}]\lambda\Delta t = R^{\Delta t}s_t.$$

Then $\mathbb{E}[\hat{J}] = \frac{(R^{\Delta t}-1)s_t - \mu(s_t,t)\Delta t - \lambda k(s_t,t)\Delta t^2}{\lambda\Delta t} = \frac{rs_t - \mu(s_t,t) + k(s_t,t)\lambda}{\lambda} + o(1)$. So the infinitesimal generator for the Jump component of the lower bound process is given by

$$\begin{aligned}
\mathcal{A}^{s_t^{\hat{J}}} f = &\lim_{\Delta t \to 0^+} \frac{\mathbb{E}[f(s_t + \hat{J}\Delta N) - f(s_t)]}{\Delta t} \\
=&\lambda\Delta t \frac{\mathbb{E}[f(s_t + \hat{J}) - f(s_t)] + O(\Delta t)}{\Delta t} \\
=&\lambda\mathbb{E}[f(s_t + \lim_{\Delta t \to 0^+} \hat{J}) - f(s_t)] + \lim_{\Delta t \to 0^+} O(\Delta t) \\
=&\lambda\mathbb{E}[f(s_t + J^l) - f(s_t)],
\end{aligned}$$

where $J^l = \lim_{\Delta t \to 0^+} \hat{J} = \lim_{\Delta t \to 0^+} \{J | J < \bar{j}\} = \{J | J < j^l\}$, where the third equality holds because $\{J | J < \bar{j}\}$ is continuous in $\bar{j}$, since J is a continuous random variable. j^l is chosen so that

$$\begin{aligned}
\mathbb{E}[J^l] =&\mathbb{E}[\lim_{\Delta t \to 0^+} \hat{J}] \\
=& \lim_{\Delta t \to 0^+} \frac{(R^{\Delta t} - 1)s_t - \mu(s_t, t)\Delta t - \lambda k(s_t, t)\Delta t^2}{\lambda\Delta t} \\
=& \lim_{\Delta t \to 0^+} \frac{rs_t - \mu(s_t, t) + k(s_t, t)\lambda}{\lambda} + o(1) \\
=&\frac{rs_t - \mu(s_t, t) + k(s_t, t)\lambda}{\lambda}
\end{aligned}$$

Therefore, the lower bound risk-neutral process is given by

$$ds^q = (\mu(s_t, t) - k(s_t, t)\lambda)dt + \sigma(s_t, t)dw_t + J^l dN.$$

So, by Ito's Lemma for Jump-Diffusion Processes, if ds_t^q is the risk-neutral process differential, and $f : \mathbb{R}^2 \to \mathbb{R}$ is some smooth function then

$$\mathbb{E}_t[f(s_t + ds_t^q, t + dt) - f(s_t, t)] = (\frac{\partial f}{\partial t} + \mu(s_t, t)\frac{\partial f}{\partial x} + \sigma(s_t, t)^2 \frac{\partial^2 f}{\partial x^2} + \lambda \mathbb{E}_t[f(s_t + J^l) - f(s_t)])dt = rf(s_t, t)dt.$$

Rearranging, one arrives at the following PDE:

$$\frac{\partial f}{\partial t} + \mu(s_t, t)\frac{\partial f}{\partial s_t} + \sigma(s_t, t)^2 \frac{\partial^2 f}{\partial s_t^2} + \lambda \mathbb{E}_t[f(s_t + J^l) - f(s_t)] - rf = 0. \quad (3.4)$$

This generates the following Theorem:

Theorem 6 (Perrakis)**.** *Fix $t < T$, let s_t be represent the Jump-Diffusion process 3.4.2, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be some smooth function, convex in the first argument, such that $f(s_T, T) = g(s_T)$ and there exists a random variable D, measurable in ds_t, such that $\mathbb{E}_t[Dds_t] = 0, \mathbb{E}_t[D] = 1 - rdt, \mathbb{E}_t[Df(s_{t+dt}, t+dt)] = f(s_t, t)$, and D is decreasing in ds_t. If $\underline{f} : \mathbb{R}^2 \to \mathbb{R}$ solves (3.4) with boundary condition $\underline{f}(s_T, T) = g(s_T)$ then $\underline{f}(s_t, t) \leq f(s_t, t)$.*

3.4.3 An Integral Representation For a Particular Class of Jump-Diffusion Processes

Consider the case where $\mu(s_t, t) = \mu s_t$, $k(s_t, t) = ks_t$, $\sigma(s_t, t) = \sigma s_t$, and $J = J_t s_t$, where J_t is independent of s_t, homogeneous in t, and J_t and J_s are independent for all $s \neq t$. Then

$$ds = (\mu - k\lambda)s_t dt + \sigma s_t dw_t + J_t s_t dN.$$

But observe that the log of this process is infinitely divisible. By [20], the characteristic function ψ of the log-process at time t takes the form

$$\psi_t(u) = \exp[t(i(\mu - k\lambda)u - \frac{\sigma^2}{2}u^2 + \int_{-\infty}^{\infty}(e^{-iux} - 1)M(dx))],$$

where M is a Levy measure such that $M(A) = \lambda\mathbb{P}[J_t \in A], \forall A \in \mathcal{F}$. Therefore the characteristic function of the lower bound risk-neutral process is

$$\psi_t^q(u) = \exp[t(i(\mu - k\lambda)u - \frac{\sigma^2}{2}u^2 + \int_{-\infty}^{\infty}(e^{-iux} - 1)M^q(dx))],$$

where $M^q = \lambda\frac{M\chi_{\cdot > j^*}}{M((j^*,\infty))}$, where j^* is chosen so that $\int_{-\infty}^{\infty} xM^q(dx) = r - \mu + k\lambda$. Therefore, by Carr and Madan[5], if ψ_1^q is analytic, the lower bound price $f^q(K)$ of a call with strike price K is

$$f^q(K) = \frac{K^{-\alpha}}{2\pi}\int_{-\infty}^{\infty} e^{-iu\ln K}\psi^\alpha(u)du,$$

where $\psi^\alpha(u) = \frac{e^{-rT}s_0^{iu}\psi_T^q(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}$ and $\alpha > 0$ is arbitrary.
Now let

$$\psi_t^h(u) = \exp[t(i(\mu - k\lambda)u - \frac{\sigma^2}{2}u^2 + \int_{-\infty}^{\infty}(e^{-iux} - 1)M(dx)) + \frac{r-\mu}{m}t(e^{-ium} - 1)],$$

where $m = \inf\{x \in \mathbb{R} : M((-\infty, x)) > 0\}$. Then, as before, if ψ_1^q is analytic then the upper bound call price f^h is given by

$$f^h(K) = \frac{K^{-\beta}}{2\pi}\int_{-\infty}^{\infty} e^{-iu\ln K}\psi^\beta(u)du,$$

where $\psi^\beta(u) = \frac{e^{-rT}s_0^{iu}\psi_T^h(u-(\beta+1)i)}{\beta^2+\beta-u^2+i(2\beta+1)u}$ and $\beta > 0$ is arbitrary. Now let p_J be the density of J_t. Then an alternative expression for ψ_t^q and ψ_t^h is

$$\psi_t^q(u) = \exp[t(i(\mu - k\lambda)u - \frac{\sigma^2}{2}u^2 + \lambda\int_{-\infty}^{\infty} e^{-iux}p_J^q(x)dx - \lambda)],$$

$$\psi_t^h(u) = \exp[t(i(\mu - k\lambda)u - \frac{\sigma^2}{2}u^2 + \lambda\int_{-\infty}^{\infty} e^{-iux}p_J(x)dx - \lambda) + \frac{r-\mu}{m}t(e^{-ium} - 1)],$$

where $p_J^q = \frac{p_J\chi_{\cdot > j^*}}{\int_{j^*}^{\infty} p_J(x)dx}$, where j^* is chosen so that $\int_{-\infty}^{\infty} xp_J^q(x)dx = r - \mu + k\lambda =: r_J$ and $m = \inf\{x \in \mathbb{R} : p_J(x) > 0\}$. If ψ_J represents the characteristic function of p_J and if g_α is some function such that

$$g_\alpha(x) := r_J\pi - r_J e^{\alpha x}\Re\left[\int_{-\infty}^{\infty} e^{-iux}\psi_J^\alpha(u)du\right] - e^x\int_{-\infty}^{\infty} e^{-iux}\phi_J(u)du,$$

where $\psi_J^\alpha = \frac{1}{iu-\alpha}\psi_J(u+i\alpha)$, $\phi_J(u) = \frac{1}{1-iu}\psi_J(u)$, and $\alpha \neq 0$, then j^* solves $g_\alpha(j^*) = 0$ if ψ_J is analytic.

3.5 Perrakis and Ryan's Bounds

Perrakis and Ryan[32] was able to find second-order stochastic dominance bounds on a call option in continuous probability space under the consumption-based model with the additional assumption that the underlying security has positive correlation with consumption, using the physical distribution of the price change as an input. It states the following theorem:

Theorem 7. *After a single revision period, if no second-order stochastic dominance arbitrage opportunities exist, the price of a call option $C(S, X, i)$ with underlying price S, strike price X, and riskless interest rate i satisfies*

$$\max\left(0, S + \frac{1}{1+i}\left[-X + \int_0^X F(\omega - S)d\omega\right]\right) \leq C(S, X, i) \leq S + \frac{S}{S + E[Y]}\left[-X + \int_0^X F(\omega - S)d\omega\right],$$

where Y is the change in price after one period and F is the CDF of Y.

Proof. Perrakis and Ryan constructs three portfolio, given an initial investment of S. The first portfolio (A) buys one share of the stock. The second portfolio (B) buys a call at price C buys $S - C$ riskless bonds. Finally, the third portfolio (C) buys S/C calls. Observe that (A) pays out $S + Y$, (B) pays out $h_1(Y) := (S - C)(1 + i) + (S + Y - X)^+$, and (C) pays out $h_2(Y) := \frac{S}{C}(S + Y - X)^+$. Since all these portfolios cost the same, this yields

$$\int_{-S}^{\infty}(S + y)Z(y)dF(y) = \int_{-S}^{\infty} h_1(y)Z(y)dF(y),$$

where Z is the pricing kernel. Subtracting from both sides, one obtains

$$\int_{-S}^{\infty}[(S + y) - h_1(y)]Z(y)dF(y) = 0.$$

Observe that, since $(S + y) - h_1(y)$ is increasing in y, there exists some constant y_1 such that $(S+y_1) - h_1(y_1)$ is 0, $(S+y) - h_1(y)$ is non-negative for all $y > y_1$, and non-positive for all $y < y_1$. Therefore, since the pricing kernel Z is non-increasing and positive,

$$0 = \int_{-S}^{\infty} [(S+y) - h_1(y)] Z(y) dF(y) \leq Z(y_1) \int_{-S}^{\infty} [(S+y) - h_1(y)] dF(y),$$

which means $\int_{-S}^{\infty} [(S+y) - h_1(y)] dF(y) \geq 0$. Rearranging terms,

$$S + \mathbb{E}[Y] \geq (S - C)(1+i) + \mathbb{E}[(S + Y - X)^+].$$

Solving for C gives the desired result. Similarly, for the upper bound:

$$\int_{-S}^{\infty} (S+y) Z(y) dF(y) = \int_{-S}^{\infty} h_2(y) Z(y) dF(y),$$

where Z is the pricing kernel. Subtracting from both sides yields

$$\int_{-S}^{\infty} [(S+y) - h_2(y)] Z(y) dF(y) = 0.$$

Observe that there exists some point, call it y_2, such that $(S+y_2) - h_2(y_2) = 0$, $(S+y) - h_2(y)$ is non-negative for all $y < y_2$, and non-positive for all $y > y_2$. Therefore, since the pricing kernel Z is non-increasing and positive,

$$0 = \int_{-S}^{\infty} [(S+y) - h_2(y)] Z(y) dF(y) \geq Z(y_2) \int_{-S}^{\infty} [(S+y) - h_2(y)] dF(y),$$

which means $\int_{-S}^{\infty} [(S+y) - h_2(y)] dF(y) \leq 0$. Rearranging terms, one arrives at

$$S + \mathbb{E}[Y] \leq \frac{S}{C} \mathbb{E}[(S + Y - X)^+].$$

Solving for C gives the desired result. □

Observe that these bounds are wider than Ritchken's bounds. However, the method used for this proof will provide a framework for deriving option bounds under different models in chapter 5.

Chapter 4

Option Bounds and Stochastic Dominance: Deterministic Volatility

This section discovers methods of implementation for Ritchken option bounds in discrete time, as well as generalizing Ritchken bounds for the case of random time.

4.1 Stochastic Dominance Option Bounds

4.1.1 Implementation of Continuous-Time No-Revision Option Bounds with Fourier Transform

Let $f(K)$ be the price of a call at strike price K with no revision opportunities. Then

$$f(K) = \int_0^\infty (s-K)^+ D(s) p_s(s) ds = \int_{-\infty}^\infty (e^x - e^k)^+ D(e^x) p_x(x) dx,$$

where $x = \ln s$, where s is the log-price of the stock, $k = \ln K$, p_s is the PDF of the stock price at expiry, p_x is the PDF of the log-stock price at expiry,

D is the pricing kernel as a function of the stock price at expiry. Now, the lower bound pricing kernel is given by $\underline{D}$ where $\underline{D}(s) = \frac{\chi_{s<s_*}}{r^T\mathbb{P}(S_T<s_*)} = \frac{\chi_{x<x_*}}{r^T\mathbb{P}(X_T<x_*)}$, where s_* is chosen so that $\int_0^\infty sD(s)p_s(s)ds = S_0$, $x_* = \ln s_*$, and $X_T = \ln S_T$. If ψ is the characteristic function of the log-stock price at expiry, and $\underline{f}(K)$ is the lower bound call price at K, then

$$\begin{aligned}\underline{f}(K) &= \int_{-\infty}^{\infty}(e^x - e^k)^+ D(e^x)p_x(x)dx \\ &= \int_k^{\infty}(e^x - e^k)D(e^x)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\psi(u)dudx \\ &= \frac{1}{2\pi r^T\mathbb{P}(X_T < x_*)}\int_{-\infty}^{\infty}\psi(u)\int_k^{\infty}(e^{x(1-iu)} - e^{k-iux})\chi_{x<x_*}dxdu.\end{aligned}$$

Now,

$$\begin{aligned}\int_k^{\infty}(e^{x(1-iu)} - e^{k-iux})\chi_{x<x_*}dx &= \int_k^{x_*} e^{x(1-iu)}dx - \int_k^{x_*} e^{k-iux}dx \\ &= \frac{e^{x_*(1-iu)} - e^{k(1-iu)}}{1-iu} - \frac{e^{k-iux_*} - e^{k(1-iu)}}{iu} \\ &= \frac{e^{x_*(1-iu)}}{1-iu} - e^k\frac{e^{-iux_*}}{iu} - e^k\frac{e^{-iuk}}{1-iu} + e^k\frac{e^{-iuk}}{iu}\end{aligned}$$

Furthermore, by the Gil-Palaez's Fourier Inversion Theorem for CDFs [13], we know that

$$\mathbb{P}(X_T < x_*) = \frac{1}{2} - \frac{1}{\pi}\Re\left[\int_0^{\infty}\frac{e^{-iux_*}\psi(u)}{iu}du\right],$$

where $\Re[z]$ is the real part of z for any $z \in \mathbb{C}$. Finally, if ϕ is some analytic complex function such that $\lim_{|\Re[z]|\to\infty}\phi(z) = 0$ then, from the Cauchy Integration Theorem, for any $\alpha, b \in (0, \infty)$,

$$\int_{-b}^{b}\phi(z)dz + \int_b^{b+i\alpha}\phi(z)dz + \int_{b+i\alpha}^{-b+\alpha i}\phi(z)dz + \int_{-b+i\alpha}^{-b}\phi(z)dz = 0.$$

Taking the limit as b goes to ∞, we see that the second and fourth terms vanish, since ϕ goes to 0 as the real part of its argument goes to ∞ or $-\infty$.

Therefore,

$$\int_{-\infty}^{\infty} \phi(z)dz = \int_{-\infty+i\alpha}^{\infty+i\alpha} \phi(z)dz.$$

If ψ is analytic, then $\phi(u) := (e^{-iux_*} - e^{-iuk})\frac{\psi(u)}{iu}$ is analytic everywhere except at 0 for all k. Moreover, since ψ is a characteristic function, it is bounded, so $\lim_{|\Re[z]|\to\infty} \phi(z) = 0$. Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty}(e^{-iux_*} - e^{-iuk})\frac{\psi(u)}{iu}du =& \int_{-\infty+i\alpha}^{\infty+i\alpha}(e^{-iux_*} - e^{-iuk})\frac{\psi(u)}{iu}du \\ =& e^{\alpha x_*}\int_{-\infty}^{\infty} e^{-iux_*}\frac{\psi(u+\alpha i)}{iu-\alpha}du - e^{\alpha k}\int_{-\infty}^{\infty} e^{-iuk}\frac{\psi(u+\alpha i)}{iu-\alpha}du.\end{aligned}$$

Now let

$$f_\alpha(K) := \frac{c + Kc_\alpha - K^{(1+\alpha)}\int_{-\infty}^{\infty} e^{-iu\ln K}\psi_\alpha(u)du - \int_{-\infty}^{\infty} e^{-iu\ln K}\phi(u)du}{d_\alpha},$$

where $c = \int_{-\infty}^{\infty}\frac{e^{x_*(1-iu)}}{1-iu}\psi(u)du$, $c_\alpha = \int_{-\infty}^{\infty}\frac{e^{\alpha x_* - iux_*}}{iu-\alpha}\psi(u+i\alpha)du$, $\psi_\alpha(u) = \frac{1}{iu-\alpha}\psi(u+i\alpha)$, $\phi(u) = \frac{1}{1-iu}\psi(u)$, and $d_\alpha = r^T\pi - r^T e^{\alpha x_*}\Re\left[\int_{-\infty}^{\infty}\frac{e^{-iux_*}\psi(u+i\alpha)}{iu-\alpha}du\right]$.
If ψ is analytic then $\underline{f} = f_\alpha$ for arbitrary $\alpha \neq 0$. If ψ is not analytic then by the Dominated Convergence Theorem, $\underline{f} = \lim_{\alpha\to 0} f_\alpha$.
Moreover, we have that

$$\begin{aligned}S_0 =& \int_0^{\infty} sD(s)p_s(s)ds \\ =& \int_{-\infty}^{\infty} e^x D(e^x)p_x(x)dx \\ =& \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^x e^{-iux}\chi_{x<x_*}dx\psi(u)du}{2\pi r^T\mathbb{P}(X_T < x_*)}.\end{aligned}$$

Taking the inner integral:

$$\int_{-\infty}^{\infty} e^x e^{-iux}\chi_{x<x_*}dx = \int_{-\infty}^{x_*} e^{(1-iu)x}dx = \frac{e^{(1-iu)x_*}}{1-iu}$$

Furthermore,

$$S_0 2\pi r \mathbb{P}(X_T < x_*) = S_0 r^T \pi - S_0 r^T \Re \left[\int_{-\infty}^{\infty} e^{-iux_*} \frac{\psi(u)}{iu} du \right].$$

Now let

$$g_\alpha(x) := S_0 r^T \pi - S_0 r^T e^{\alpha x} \Re \left[\int_{-\infty}^{\infty} e^{-iux} \psi_\alpha(u) du \right] - e^x \int_{-\infty}^{\infty} e^{-iux} \phi(u) du.$$

Now let $g := \lim_{\alpha \to 0^+} g_\alpha$. If ψ is analytic then $g_\alpha = g$ for all $\alpha \neq 0$. x_* must be chosen so that $g(x_*) = 0$.
When $\inf S = 0$, the upper bound price $\overline{f}(K)$ at strike price K is given by

$$\overline{f}(K) = \frac{S_0 - r^{-T} S_T^{min}}{\mathbb{E}[S_T] - S_T^{min}} \mathbb{E}[C_T(S_T)] + \frac{r^{-T}\mathbb{E}[S_T] - S_0}{\mathbb{E}[S_T] - S_T^{min}} (S_{min} - K)^+,$$

where $S_T^{min} = \inf\{x : \mathbb{P}[S_T \geq x] > 0\}$. Thanks to the work of Carr and Madan[5] and the fact that $\mathbb{E}[S_T] = \psi(-i)$, we have

$$\overline{f}(K) = \frac{K^{-\alpha}(S_0 - r^{-T} S_T^{min})}{2\pi(\psi(-i) - S_T^{min})} \int_{-\infty}^{\infty} e^{-iu \ln K} \overline{\psi}_\alpha(u) du + \frac{r^{-T}\psi(-i) - S_0}{\psi(-i) - S_T^{min}} (S_{min} - K)^+,$$

where $\overline{\psi}_\alpha(u) = \frac{\psi(u - (\alpha+1)i)}{\alpha^2 + \alpha - u^2 + i(2\alpha+1)u}$ (not the complex conjugate) and $\alpha > 0$ is arbitrary. Figure (4.1) shows the result of using fft to find upper and lower bounds are option prices. The script can be found in Appendix section (8.2).

4.1.2 Option Bounds with and without Distributional Assumptions

The option bounding method described in section 3.3.6 requires us to compute a lattice with corresponding probabilities. In this paper, two methods of constructing such a lattice will be discussed.

1. **Quantile Method (see Figure 4.2):** A lattice is constructed by simply taking the quantiles of the data. This method has the advantage of being totally non-parametric, making no distributional

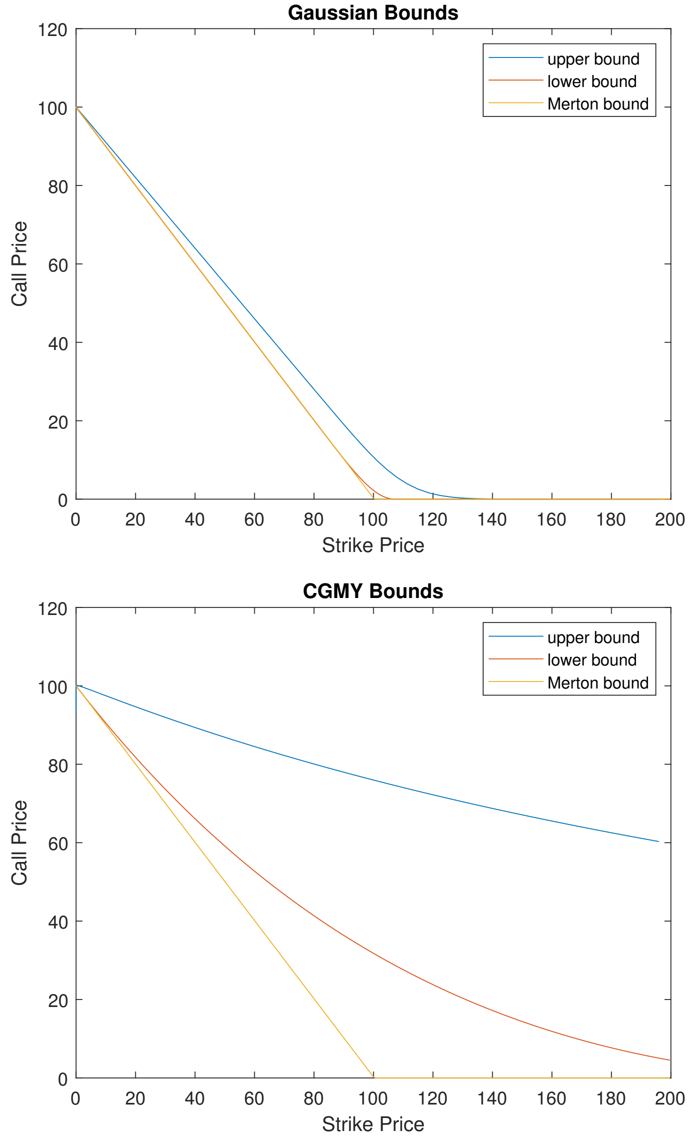


Figure 4.1: Using the fft method in this section, Ritchken's option bounds are found for prices on calls with various strike prices with underlying price 100, riskless rate 3%, and time to expiry one month. Top: Continous-space option bounds when the log-price of the underlying is distributed according to normal random variable with mean 4.705 and variance 0.01. Bottom: Continous-space option bounds when the log-price of the underlying is distributed according to CGMY random variable with $C = .02, G = .03, Y = 1.7, M = .02, m = 4.705$. These bounds are compared with Merton's bound: $(S - Ke^{-rT})^+$

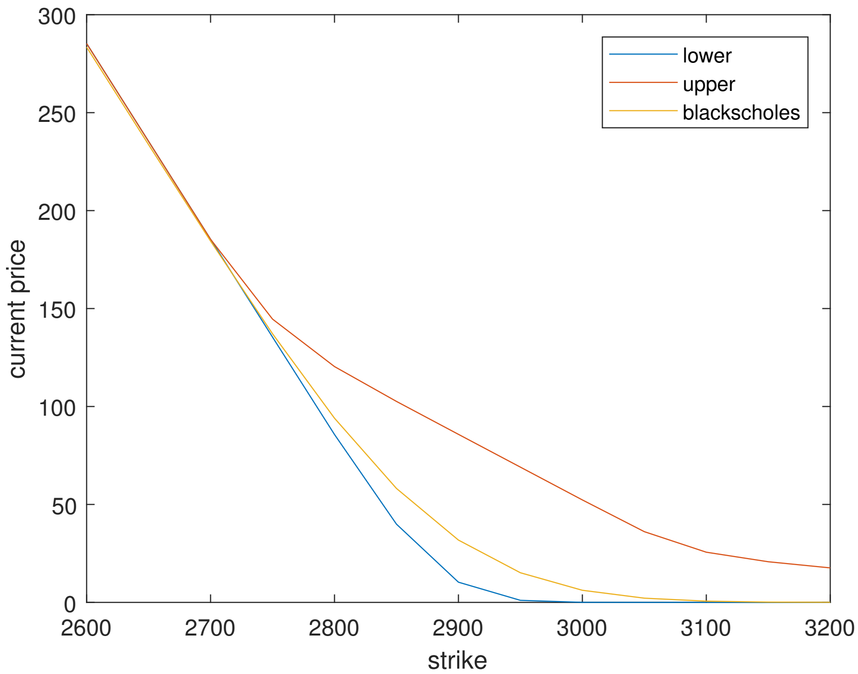


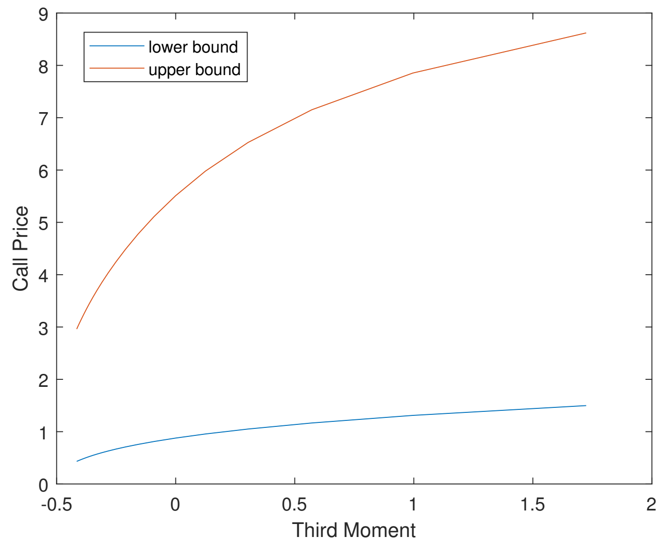
Figure 4.2: Bounds on calls on the S&P500 of various strike prices when underlying price is 2883.2, the annual riskless rate is 3.33%, and time to maturity is 9 days. The quantile method for 6 nodes is used for lattice estimation.

assumptions about the data whatsoever. However, this method can fail to capture very fat tails when few quantiles are taken.

2. **Moments-Matching Method (see Figure 4.3):** The second method chooses a lattice that matches the statistical moments of the underlying distribution. In particular, a lattice for a CGMY process is chosen as the underlying distribution. The maximum entropy solution with constraint that the first n moments match those of the CGMY distribution, utilizing Rajan et al.'s[27] algorithm, are chosen for generating the lattice. This method assumes all moments are finite. However, this method also captures fat tails more effectively than the quantile method, as it can match higher-order even moments.

4.1.3 Option Bounds with Random Revision Times

We now consider bounding option prices when the revision times are random. Let each period have a random but finite number of possible revision



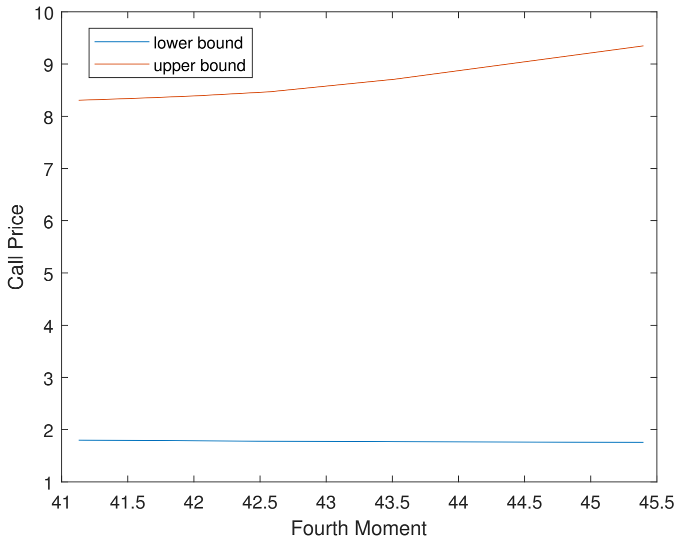


Figure 4.3: Left: Bounds for the price of a call on a theoretical stock with underlying price of 10, strike price of 20, riskless rate of 3%, and 1 revision Opportunity. The lattice is chosen to match the first 5 moments of a CGMY distribution with Y=0.8 with the C,G, and M parameters varying so that variance remains constant as kurtosis increases. Right: Bounds for the price of a call on a theoretical stock with underlying price of 10, strike price of 20, riskless rate of 3%, and 1 revision Opportunity. The lattice is chosen to match the first 5 moments of a geometric CGMY distribution with Y=0.8, C=2, and G=2, with M varying so that skewness varies.

opportunities $\tau \in \{0, 1, ..., m\}$. The solution for the bounds can be derived by the following argument: Let D be our pricing kernel. Then the current price of our call C_0 is given by $C_0 = r^{-1}E[DC_1]$, where C_1 is the price of the call in the next period. However, if our revision times are random, given by τ, and C_τ is a subordinated process denoting the price of our call in the next period then $C_0 = E[DC_\tau] = E[E[DC_\tau|\tau]] = \sum_{j=1}^{m} E[DC_j|\tau = j]$. This holds for any D in the feasible region, including the D's that produce the bounds. So let $U \in \{u_1, ..., u_n\}$ denote the random geometric change in the stock price after one revision. Let $\delta_i := P[\tau = i]$, $\pi_i(k) := P[U = u_i|\tau = k]$, let r denote the riskless discount factor and let S_0 denote the current price of the underlying. Then the upper bound $\overline{C}$ and lower bound $\underline{C}$ on a call with strike price X is given by

$$\overline{C} = r^{-1}\{\sum_{\ell=1}^{m} \sum_{\{j_\mu\}_\ell} \frac{\ell!}{j_1!...j_n!} p_{\ell,1}^{j_1}...p_{\ell,h+1}^{j_{h+1}} \delta_\ell \max(0, u^{j_1}...u^{j_n} S_0 - X)\}, \tag{4.1}$$

$$\underline{C} = r^{-1}\{\sum_{\ell=1}^{m} \sum_{\{j_\mu\}_\ell} \frac{\ell!}{j_1!...j_n!} q_{\ell,1}^{j_1}...q_{\ell,h+1}^{j_{h+1}} \delta_\ell \max(0, u^{j_1}...u^{j_n} S_0 - X)\}, \tag{4.2}$$

where $p_1 = \hat{\theta}\pi_1(\ell) + (1 - \hat{\theta}), p_j = \hat{\theta}\pi_i(\ell)$ for $j = 2, ..., n$, $\ell = 1, ..., m$ and $q_{\ell,h+1} = \hat{\beta}, q_{\ell,j} = (1 - \hat{\beta})\frac{\pi_j(\ell)}{\sum_{i=1}^{h} \pi_j(\ell)}$ for $j = 1, ..., h$, $\ell = 1, ..., m$, where $\{j_\mu\}_\ell$ is the collection of sets of non-negative integers such that $\sum_{\mu=1}^{n} j_\mu = \ell$ for (4.1) and $\sum_{\mu=1}^{h+1} j_\mu = \ell$ for (4.2), where $\hat{\theta} = \frac{r-u_1}{E[U]-u_1}$ and $\hat{\beta} = \frac{r-E[U|U<u_h]}{u_{h+1}-E[U|U<u_h]}$, where $h = \max\{j : u_j < r\}$.

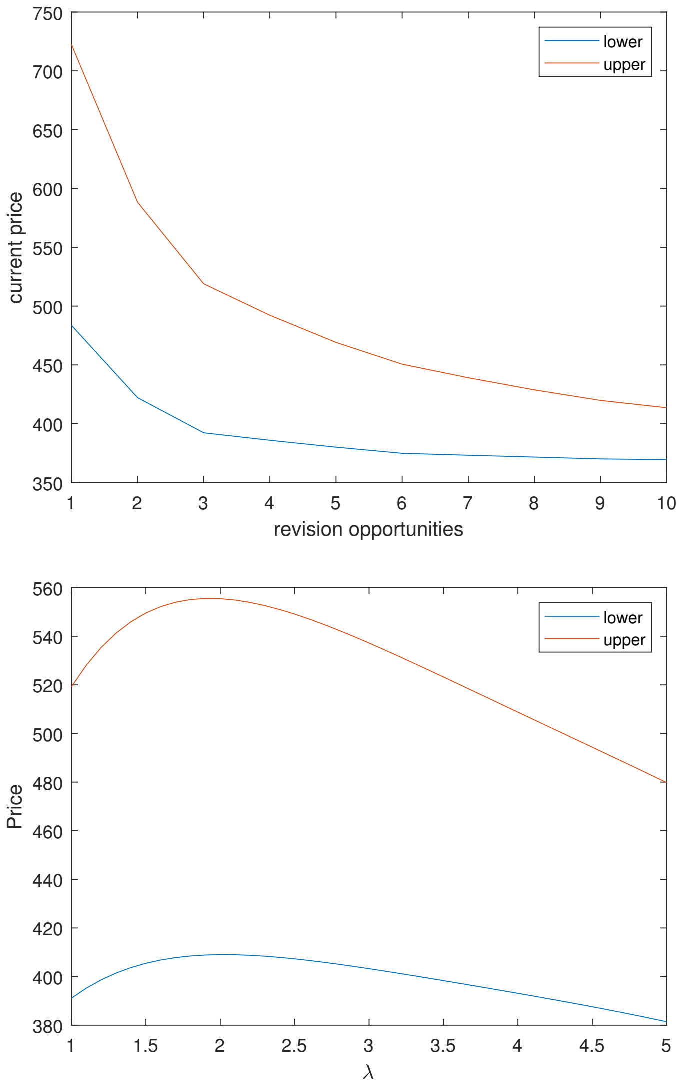


Figure 4.4: Left: Upper and lower bound call prices with varying revision opportunities. Right: Upper and lower bounds values for Poisson distributed random revision time for varying values of the λ parameter. For both graphs, the lattice was generated by a quantile method on a geometric CGMY process with $C = 0.5, G = 2, M = 3.5, Y = 0.5$ and drift 0.1. The underlying price is 2883, the strike is 2600, the riskless rate is 3.33%, and the time to expiry is 1. The MATLAB script for this can be found in Appendix 8.1

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Chapter 5

Option Bounds and Stochastic Dominance: Stochastic Volatility

This chapter uses concepts of stochastic dominance to discover option bounds in the context of continuous-time models with stochastic volatility.

5.1 Option Bounds on the Heston's Stochastic Volatility Model

For the sake of this bounding procedure, we make all the same assumptions as Perrakis and Ryan[32] with three additional assumptions: (i) $\mathrm{corr}(ds, dv) \le 0$, (ii) $\mathrm{corr}(dv, dc) \le 0$, and (iii) $\mathrm{corr}(ds, dc) \ge \mathrm{corr}(dv, dc)\mathrm{corr}(ds, dv)$. Next, we will utilize the fact that, under the Heston model, a call option always has a positive vega (partial derivative with respect to volatility) due to the following lemma:

Lemma 3. *Let s and v be stochastic processes whose movement is described by* (2.2) *and* (2.1)*, respectively, and let $C(t, v(t), s(t))$ be the price of a European call at time $t \in [0, T]$ with underlying price $s(t)$, volatility $v(t)$, and expiry date T. If no arbitrage opportunities exist then $\frac{\partial C}{\partial v} \ge 0$.*

Proof. Let $C_{BS}(s(t), v(t), t)$ denote the Black-Scholes price of a call at time

t with underlying price $s(t)$ and implied volatility $v(t)$. Let v denote the volatility process described by 2.1 with initial volatility $v(0) = v_0$ and, for any $\delta \geq 0$, let s^δ and v^δ denote the same stock and volatility process, receptively, with initial volatility $v(0) = v_0 + \delta$. Observe that if, for any t, $v^\delta(t) < v(t)$, then, by continuity of an Ito process, there exists $t_0 < t$ such that $v(t_0) = v^\delta(t_0)$. By the Markov property of v, this implies $v(t) = v^\delta(t)$ for all $t \geq t_0$ almost surely, which is a contradiction. Therefore $v^\delta(t) \geq v(t)$ for all t almost surely. Now, if r is the instantaneous riskless rate, k is the strike price, $C(s(t), v(t), t)$ denotes the actual price of a call at time t with underlying price $s(t)$ and volatility $v(t)$, and no arbitrage opportunities exist then there exists an equivalent measure $\mathbb{Q}$ such that

$$\begin{aligned}
C(s(0), v_0 + \delta, 0) =& e^{-rT} E^{\mathbb{Q}}[(s^\delta(T) - k)^+] \\
=& E^{\mathbb{Q}}[e^{-rT} E^{\mathbb{Q}}[(s^\delta(T) - k)^+ | v]] \\
=& E^{\mathbb{Q}}[C_{BS}(s(0), \frac{1}{T}\int_0^T v^\delta(s) ds, 0)] \\
\geq& E^{\mathbb{Q}}[C_{BS}(s(0), \frac{1}{T}\int_0^T v(s) ds, 0)] \\
=& E^{\mathbb{Q}}[e^{-rT} E^{\mathbb{Q}}[(s(T) - k)^+ | v]] \\
=& e^{-rT} E^{\mathbb{Q}}[(s(T) - k)^+] \\
=& C(s(0), v_0, 0),
\end{aligned}$$

where the inequality holds because C_{BS} is non-decreasing in implied volatility and $\frac{1}{T}\int_0^T v^\delta(s) ds \geq \frac{1}{T}\int_0^T v(s) ds$ because $v^\delta(t) \geq v(t)$ for all $t \geq 0$ almost surely. □

Now consider a portfolio P_1 where we sell a call C and buy $\frac{\partial C}{\partial s}$ units of the underlying. Then it follows from Ito's lemma that

$$dP_1 = m_1 dt + \omega_1 dz_2(t), \tag{5.1}$$

where

$$m_1 = -\frac{\partial C}{\partial t} - \frac{\partial C}{\partial v}\kappa(\theta - v) - \frac{1}{2}\frac{\partial^2 C}{\partial s^2} v s^2 - \frac{\partial^2 C}{\partial s \partial v}\rho\sigma v s - \frac{1}{2}\frac{\partial^2 C}{\partial v^2}\sigma^2 v \tag{5.2}$$

and

$$\omega_1 = -\frac{\partial C}{\partial v}\sigma\sqrt{v} \tag{5.3}$$

Now, since $\frac{\partial C}{\partial v} > 0$, ω_1 is negative and so, by assumption (ii), $\mathrm{corr}(dP_1, dc) \geq 0$. We therefore require $m_1 \geq rP_1 = r\frac{\partial C}{\partial s}s - rC$, where the lower bound on C is given in the equality case. Consequently, the lower bound on our call solves the following PDE:

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial v}\kappa(\theta - v) + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}vs^2 + \frac{\partial^2 C}{\partial s \partial v}\rho\sigma^2 vs + \frac{1}{2}\frac{\partial^2 C}{\partial v^2}\sigma v + r\frac{\partial C}{\partial s}s - rC = 0 \tag{5.4}$$

with boundary conditions

$$C(s(T), v(T), T) = (s - k)^+ \tag{5.5}$$

$$C(0, v, t) = 0 \tag{5.6}$$

$$\lim_{s\to\infty}\frac{\partial C}{\partial s} = 1, \tag{5.7}$$

where k is the strike price.
For the next part, we utilize the following decomposition:

$$dz_1(t) = \rho dz_2(t) + \sqrt{1-\rho^2}dz_3(t), \tag{5.8}$$

where $dz_3(t)$ is a Wiener process independent of $dz_1(t)$. As a result of assumption (iii), $\mathrm{corr}(dz_3, dc) \geq 0$. Now construct a portfolio P_2 where we buy one call and sell $\frac{\partial C}{\partial s} + \frac{\sigma}{\rho s}\frac{\partial C}{\partial v}$ units of the underlying. Then, by Ito's lemma,

$$dP_2 = m_2 dt + \omega_2 dz_3(t) \tag{5.9}$$

where

$$m_2 = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial v}\kappa(\theta - v) + \frac{1}{2}\frac{\partial^2 C}{\partial s^2} + \frac{\partial^2 C}{\partial s \partial v}\rho\sigma vs + \frac{1}{2}\frac{\partial^2 C}{\partial v^2}\sigma^2 v - \frac{\partial C}{\partial v}\frac{\mu\sigma}{\rho} \tag{5.10}$$

and

$$\omega_2 = -\frac{\partial C}{\partial v}\frac{\sqrt{1-\rho^2}}{\rho}\sigma\sqrt{v}. \tag{5.11}$$

$\omega_2 > 0$, so $\mathrm{corr}(dP_2, dc) \geq 0$, and so we have the inequality $m_2 \geq rP_2 = rC - r\frac{\partial C}{\partial s}s - \frac{r\sigma}{\rho}\frac{\partial C}{\partial v}$, where the upper bound on C is given in the equality case. Therefore, the upper bound for C solves the following PDE:

$$\frac{\partial C}{\partial t}+\frac{\partial C}{\partial v}\kappa(\hat{\theta}-v)+\frac{1}{2}\frac{\partial^2 C}{\partial s^2}+\frac{\partial^2 C}{\partial s\partial v}\rho\sigma vs+\frac{1}{2}\frac{\partial^2 C}{\partial v^2}\sigma^2 v+r\frac{\partial C}{\partial s}s-rC = 0, \quad (5.12)$$

where $\hat{\theta} = \theta - \frac{(\mu-r)\sigma}{\kappa\rho}$, with boundary conditions

$$C(s(T), v(T), T) = (s-k)^+ \quad (5.13)$$

$$C(0, v, t) = 0 \quad (5.14)$$

$$\lim_{s\to\infty} \frac{\partial C}{\partial s} = 1, \quad (5.15)$$

where k is the strike price. Observe that the lower upper bounds equation only differs from the upper bounds equation by a change of parameter for θ. Now, let $H(s(t), v(t), t; \kappa, \theta, \rho, \sigma, k, T, r, \lambda)$ denote the Heston price at time t, where κ denotes the reversion speed, θ denotes the long-term mean, ρ denotes the correlation between the volatility and stock price, σ^2 denotes the instantaneous volatility of the volatility, k is the strike price, T is the time to maturity, r is the riskless rate, and λ represents the relative risk-aversion of the marginal investor. Observe that the upper and lower bounds PDEs are simply Heston's PDE with $\lambda = 0$. Therefore, the lower bound price is

$$H(s(t), v(t), t; \kappa, \theta, \rho, \sigma, k, T, r, 0)$$

and the upper bound price is

$$H(s(t), v(t), t; \kappa, \theta - \frac{(\mu-r)\rho\sigma}{\kappa}, \rho, \sigma, k, T, r, 0).$$

Under the Heston price, the risk-neutral dynamics of the stock is described by

$$ds = rdt + \sqrt{v}dz_1,$$

$$dv = \kappa_\lambda(\theta_\lambda - v)dt + \sigma\sqrt{v}dz_2,$$

$$dz_1 dz_2 = \rho dt,$$

where $\theta_\lambda = \frac{\kappa\theta}{\kappa+\lambda}$, $\kappa_\lambda = \kappa + \lambda$. In this paper, the upper bounds $(\overline{ds}, \overline{dv})$ and lower bound ($\underline{ds}, \underline{dv}$) price dynamics are given by

$$\overline{ds} = rdt + \sqrt{v}dz_1,$$

$$\overline{dv} = \kappa(\theta_\mu - v)dt + \sigma\sqrt{v}dz_2,$$

$$\underline{ds} = rdt + \sqrt{v}dz_1,$$

$$\underline{dv} = \kappa(\theta - v)dt + \sigma\sqrt{v}dz_2,$$

$$dz_1 dz_2 = \rho dt,$$

where $\theta_\mu = \theta - \frac{(\mu - r)\sigma}{\kappa\rho}$.

5.2 Sharpness of Bounds

We wish to show that the bounds given are sharp by showing that they are valid option prices given the assumptions. If $\text{corr}(dc, dv) = 0$ then the assumptions are not violated and P_1 has no correlation with the market. Therefore, $\mathbb{E}[dP_1] = m_1 dt = rP_1 dt$ and the lower bound follows. Alternatively, if $\text{corr}(dc, dv) = \text{corr}(dc, ds)\text{corr}(ds, dv)$ then the assumptions are not violated and P_2 has no correlation with the market. Therefore, $\mathbb{E}[dP_2] = m_2 dt = rP_2 dt$ and the upper bound follows.

5.3 Example

In this section, parameters are estimated by MLE, using Ford stock data. Then call price bounds and corresponding implied volatility bounds are found, given the estimated parameters. The following parameters are obtained: $\mu = 0.1044, \theta = 0.1205, \kappa = 3.798, \sigma = 0.3161, \rho = -0.0971$. Figure 5.1 displays the call price bounds and implied volatility bounds.

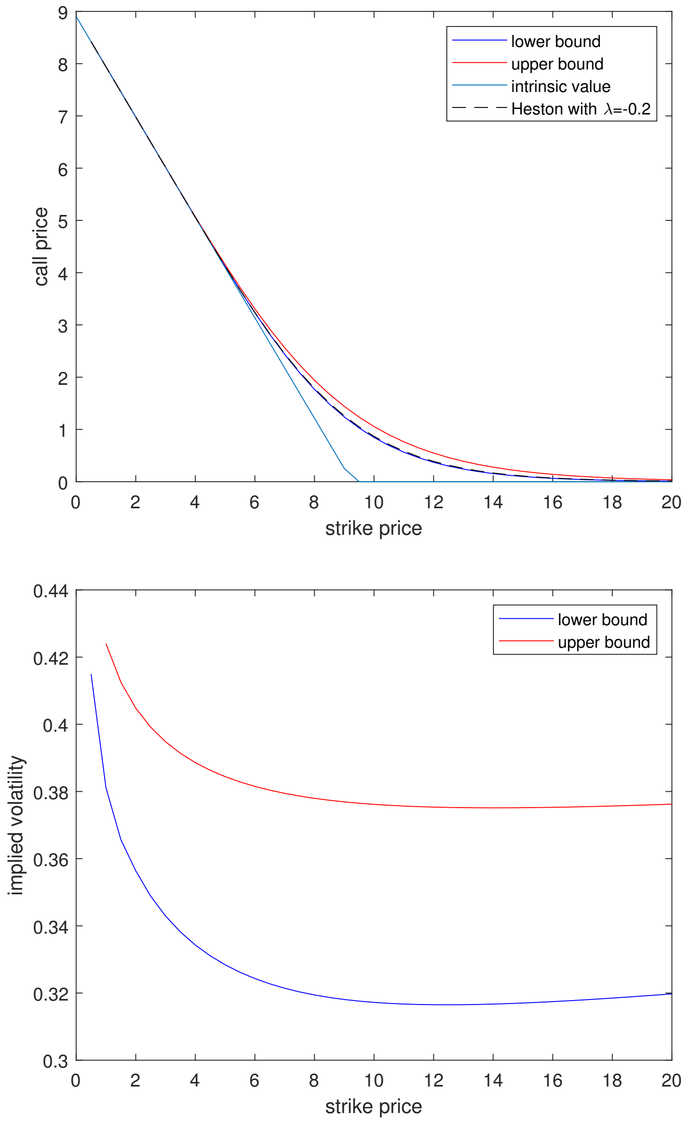

Figure 5.1: Option bounds on a Ford stock with spot price \$8.9, riskless rate 4%, and 1 year to expiry. Parameters are estimated via MLE.

5.4 Generalized Stochastic Volatility Option Bounds

This section will generalize the results from the previous section. Assume the stock $s(t)$ and volatility $v(t)$ at time t have the following differentials:

$$ds(t) = \mu(s)dt + \sigma(s)\sqrt{v}dz_1(t) \tag{5.16}$$

$$dv(t) = \alpha(v)dt + \beta(v)dz_2(t), \tag{5.17}$$

where z_1, z_2 are Wiener processes and $\beta(v) \geq 0$ for all v and $\sigma(s) > 0$ for all s. Furthermore, let $c(t)$ represent consumption at time t, let z_3 and z_4 be processes such that $dz_3 := dc - \mathbb{E}[dc]$, and $dz_4 = \frac{dz_1 - \rho_{12}dz_2}{\sqrt{1-\rho_{12}^2}}$. Note that z_3 need not be a Wiener process and note that z_4 is a Wiener process independent of s. Now let $\rho_{ij} = \frac{\text{cov}(dz_i, dz_j)}{dt}$ for $i, j = 1, 2, 3, 4$. Then any function of s, v, and t – call it f – will, by Ito's Lemma, have the following differential:

$$\begin{aligned} df(t,s,v) =& (\frac{\partial f}{\partial t} + \frac{\partial f}{\partial s}\mu(s) + \frac{\partial f}{\partial v}\alpha(v) + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma(s)^2 v + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\beta(v)^2 + \frac{\partial^2 f}{\partial v \partial s}\rho_{12}\sigma(s)\sqrt{v}\beta(v))dt \\ &+ \frac{\partial f}{\partial s}\sigma(s)\sqrt{v}dz_1(t) + \frac{\partial f}{\partial v}\beta(v)dz_2(t). \end{aligned}$$

Now, with some loss of generality, assume the following: (i) $\rho_{12} \leq 0$, (ii) $\rho_{23} \leq 0$, and (iii) $\rho_{13} \geq \rho_{23}\rho_{12}$. Observe that we do not require the market correlations to be constant. We only require them to satisfy some very fundamental and intuitive inequalities. Furthermore, assume $f(T, \cdot, v(T))$ is convex for some $T > t$ so that $\frac{\partial f}{\partial v}$ is non-negative. Then if we assemble a portfolio p_1 in which we buy a unit of f and hedge it with $\frac{\partial f}{\partial s}$ units of the stock then the differential of the portfolio will be

$$\begin{aligned} dp_1 =& df - \frac{\partial f}{\partial s}ds \\ =& (\frac{\partial f}{\partial t} + \frac{\partial f}{\partial v}\alpha(v) + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma(s)^2 v + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\beta(v)^2 + \frac{\partial^2 f}{\partial v \partial s}\rho_{12}\sigma(s)\sqrt{v}\beta(v))dt \\ &+ \frac{\partial f}{\partial v}\beta(v)dz_2(t). \end{aligned}$$

Since $\rho_{23} \leq 0$, this portfolio must have an expected return that is below the riskless rate. Consequently, a lower bound on f must satisfy the following

PDE:

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial v}\alpha(v)+\frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma(s)^2 v+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\beta(v)^2+\frac{\partial^2 f}{\partial v\partial s}\rho_{12}\sigma(s)\sqrt{v}\beta(v)-rf+r\frac{\partial f}{\partial s}s=0.$$

Alternatively, construct a new portfolio p_2 by buying one unit of f and selling $\frac{\partial f}{\partial s}+\frac{\partial f}{\partial v}\frac{\beta(v)}{\sigma(s)\rho_{12}\sqrt{v}}$ units of the stock. Then

$$\begin{aligned}
dp_2 =& df-\frac{\partial f}{\partial s}ds-\frac{\partial f}{\partial v}\frac{\beta(v)}{\sigma(s)\rho_{12}\sqrt{v}}ds\\
=&\left(\frac{\partial f}{\partial t}+\frac{\partial f}{\partial v}(\alpha(v)-\frac{\beta(v)\mu(s)}{\sigma(s)\rho_{12}\sqrt{v}})+\frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma(s)^2 v+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\beta(v)^2+\frac{\partial^2 f}{\partial v\partial s}\rho_{12}\sigma(s)\sqrt{v}\beta(v)\right)dt\\
&-\frac{\partial f}{\partial v}\beta(v)\frac{\sqrt{1-\rho_{12}^2}}{\rho_{12}}dz_4(t).
\end{aligned}$$

Since $\rho_{13}\geq\rho_{23}\rho_{12}$, we have that $\rho_{34}\geq 0$ and so dp_2 must have an expected return below the riskless rate and so the upper bound on f must satisfy:

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial v}(\alpha(v)-\frac{\beta(v)(\mu(s)-rs)}{\sigma(s)\rho_{12}\sqrt{v}})+\frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma(s)^2 v+\frac{1}{2}\frac{\partial^2 f}{\partial v^2}\beta(v)^2+\frac{\partial^2 f}{\partial v\partial s}\rho_{12}\sigma(s)\sqrt{v}\beta(v)-rf+r\frac{\partial f}{\partial s}s=0.$$

The risk neutral processes for the upper bound price (s^*,v^*) and lower bound price (s_*,v_*) are

$$ds^*=rsdt+\sigma(s)\sqrt{v}dz_1$$

$$dv^*=\alpha(v)dt+\beta(v)dz_2$$

$$ds_*=rsdt+\sigma(s)\sqrt{v}dz_1$$

$$dv_*=\left(\alpha(v)-\frac{\beta(v)(\mu(s)-rs)}{\sigma(s)\rho_{12}\sqrt{v}}\right)dt+\beta(v)dz_2.$$

5.4.1 Example: The 3/2 model

In the case of the 3/2 model, we have $\mu(s)=\mu s,\sigma(s)=s,\alpha(v)=\kappa(\theta v-v^2),\beta(v)=\eta v^{3/2},\rho_{12}=\rho,s(0)=s_0,v(0)=v_0$. The risk neutral dynamic

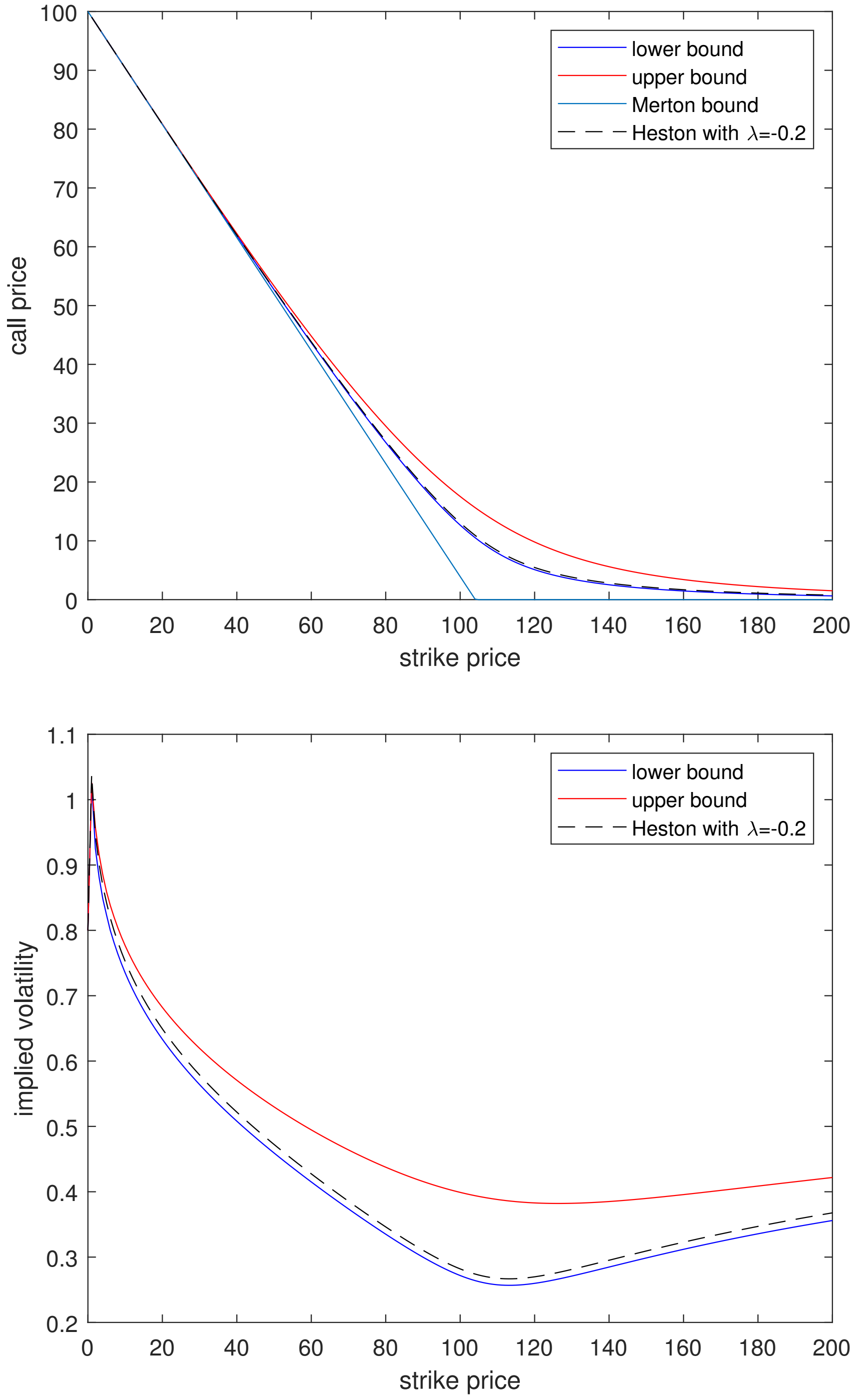

Figure 5.2: Call price and implied volatility bounds for $s = 100, v = 0.1, t = 1, \theta = 0.2, \kappa = 0.3, \sigma = 1, \rho = -0.3, r = 0.04, \mu = 0.1$

for the upper (s^*, v^*) and lower (s_*, v_*) bounds are given by:

$$ds^* = rsdt + s\sqrt{v}dz_1$$

$$dv^* = \kappa(\theta v - v^2)dt + \eta v^{3/2}dz_2$$

$$ds_* = rsdt + sdz_1$$

$$dv_* = \kappa((\theta - \frac{\eta(\mu - r)}{\rho_{12}\kappa})v - v^2)dt + \eta v^{3/2}dz_2.$$

Therefore, the upper ϕ^* and lower ϕ_* bound characteristic functions at time T are given by

$$\phi^*(u) = \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \left[\frac{2\kappa\hat{\theta}}{\eta^2 v_0(e^{\kappa\hat{\theta}T} - 1)}\right]^\alpha M\left(\alpha, \gamma, -\frac{2\kappa\hat{\theta}}{\eta^2 v_0(e^{\kappa\hat{\theta}T} - 1)}\right) s_0 e^{rT},$$

$$\phi_*(u) = \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \left[\frac{2\kappa\theta}{\eta^2 v_0(e^{\kappa\theta T} - 1)}\right]^\alpha M\left(\alpha, \gamma, -\frac{2\kappa\theta}{\eta^2 v_0(e^{\kappa\theta T} - 1)}\right) s_0 e^{rT},$$

$$\alpha = -(\frac{1}{2} - \frac{p}{\eta^2}) + \sqrt{(\frac{1}{2} - \frac{p}{\eta^2})^2 + \frac{2q}{\eta^2}},$$

$$\gamma = 2(\alpha + 1 - \frac{p^2}{\eta}),$$

$$p = -\kappa + i\eta\rho u,$$

$$\frac{iu}{2} + \frac{u^2}{2},$$

$$\hat{\theta} = \theta - \frac{\eta(\mu - r)}{\rho\kappa}$$

where Γ is the gamma function and M is the confluent hypergeometric function. So if our goal is to find the price of a vanilla European call option then we simply utilize the formula from Carr and Madan (1999) to obtain upper $f^*(k)$ and lower $f_*(k)$ bound call prices at strike price k and time to expiry T:

$$f^*(k) = \frac{e^{-rT}}{\pi} \int_0^\infty k^{-iuk-\beta} \frac{\phi^*(u)}{\beta^2 + \beta - u^2 + i(2\beta + 1)u} du,$$

$$f_*(k) = \frac{e^{-rT}}{\pi} \int_0^\infty k^{-iuk-\beta} \frac{\phi_*(u)}{\beta^2 + \beta - u^2 + i(2\beta+1)u} du,$$

where $\beta > 0$ is some free parameter that can be chosen arbitrarily (though, in practice, not every choice of β will be numerically stable) since the underlying process has finite moments.

Chapter 6

The CRRA Model

This chapter discovers an efficient method for computing option prices under the CRRA model. Various numerical examples under different physical processes demonstrate that a broad range of option price and implied volatility curves are possible under the CRRA model.

6.1 FFT Option Pricing with the CRRA Model

Theorem 8. *Under the CRRA model, if $C(k)$ is the price of a call on an index with strike price k, S_0 is the current price of the underlying, T is the time to expiry, S_T is the price of the underlying at expiration, r is the continuously compounding interest rate, and ψ is the characteristic function of $\ln \frac{S_T}{S_0}$ then, if for some $b > 0$, $\psi(u)$ is analytical on $0 < \Im[u] < b$, then*

$$C(k) = \frac{1}{2\pi\psi((\gamma-1)i)} \int_{-\infty}^{\infty} k^{-\omega i-\alpha} \frac{S_0^{1+\alpha+\omega i}\psi((\gamma-\omega i-\alpha-1)i)}{\alpha^2+\alpha-\omega^2+(2\alpha+1)\omega i} d\omega,$$

where γ is chosen so that $\frac{\psi((\gamma-1)i)}{\psi(\gamma i)} = e^{rT}$ and $\alpha \in (0, b)$.

Proof. Let ϕ_1 denote the Mellin Transform of C, let q denote the risk-neutral density, and let ϕ_2 denote the Mellin Transform of the physical density f. Then

$$\begin{aligned}\phi_1(u) =& \int_0^\infty k^{u-1} C(k) dk \\ =& \int_0^\infty k^{u-1} \int_0^\infty e^{-rT}(s-k)^+ q(s) ds dk \\ =& \int_0^\infty k^{u-1} \int_k^\infty e^{-rT}(s-k) q(s) ds dk \\ =& \int_0^\infty e^{-rT} q(s) \int_0^s k^{u-1}(s-k) dk ds \\ =& \begin{cases} \int_0^\infty e^{-rT} \frac{s^{u+1}}{u^2+u} q(s) ds, & u \neq 0 \\ \infty, & u = 0 \end{cases}.\end{aligned}$$

Now, under the CRRA model, $q(x) = \frac{x^{-\gamma} f(x)}{\int_0^\infty y^{-\gamma} f(y) dy} = \frac{x^{-\gamma} f(x)}{\phi_2(1-\gamma)}$, and so, for $u \neq 0$,

$$\phi_1(u) = \int_0^\infty e^{-rT} \frac{s^{u+1}}{u^2+u} q(s) ds = \frac{e^{-rT}}{(u^2+u)\phi_2(1-\gamma)} \int_0^\infty s^{u-\gamma+1} f(s) ds = \frac{e^{-rT}\phi_2(2-\gamma+u)}{(u^2+u)\phi_2(1-\gamma)}.$$

Now, let ψ be the characteristic function of $\ln \frac{S_T}{S_0}$. If g is the physical density of $\ln \frac{S_T}{S_0}$ then
$g(x) = \frac{d}{dx}\mathbb{P}[\ln \frac{S_T}{S_0} < x] = \frac{d}{dx}\mathbb{P}[S_T < S_0 e^x] = S_0 e^x f(S_0 e^x)$, and so

$$\begin{aligned}\phi_2(u) =& \int_0^\infty x^{u-1} f(x) dx \\ =& \int_{-\infty}^\infty (S_0 e^y)^u f(S_0 e^y) dy \\ =& S_0^{u-1} \int_{-\infty}^\infty e^{y(u-1)} g(y) dy \\ =& S_0^{u-1} \psi(-i(u-1)).\end{aligned}$$

So

$$\phi_1(u) = \frac{e^{-rT}\phi_2(2-\gamma+u)}{(u^2+u)\phi_2(1-\gamma)} = \frac{e^{-rT} S_0^{u+1} \psi((\gamma-u-1)i)}{(u^2+u)\psi(\gamma i)}.$$

Now, observe that γ must be chosen so that

$$S_0 = e^{-rT}\int_0^\infty sq(s)ds = \frac{e^{-rT}\int_0^\infty s^{1-\gamma}f(s)ds}{\phi_2(1-\gamma)} = e^{-rT}\frac{\phi_2(2-\gamma)}{\phi_2(1-\gamma)} = e^{-rT}S_0\frac{\psi((\gamma-1)i)}{\psi(\gamma i)},$$

and so $\frac{\psi((\gamma-1)i)}{\psi(\gamma i)} = e^{rT}$. Finally, by the Mellin Inversion Theorem if $\frac{S_0^{u+1}\psi((\gamma-u-1)i)}{u^2+u}$ is analytic on the strip $0 < \Im[u] < b$ for some $b > 0$,

$$\begin{aligned} C(k) =& \frac{-ie^{-rT}}{2\pi\psi(\gamma i)}\int_{\alpha-i\infty}^{\alpha+i\infty} k^{-u}\frac{S_0^{u+1}\psi((\gamma-u-1)i)}{u^2+u}du \\ =& \frac{1}{2\pi\psi((\gamma-1)i)}\int_{-\infty}^{\infty} k^{-\omega i-\alpha}\frac{S_0^{\omega i+\alpha+1}\psi((\gamma-\omega i-\alpha-1)i)}{\alpha^2+\alpha-\omega^2+(2\alpha+1)\omega i}d\omega, \end{aligned}$$

where $\alpha \in (0, b)$, since $\frac{S_0^{u+1}\psi((\gamma-u-1)i)}{u^2+u}$ is analytic on the strip $0 < \Re[u] < b$ if ψ is analytic on $0 < \Im[u] < b$. □

We can use the theorem above to design an algorithm that quickly evaluates the price of a call on an index under the CRRA model. The Fast Fourier Transform (FFT) algorithm uses Discrete Fourier Transform (DFT) so that if we have a truncated discretized characteristic function on $[-M, M]$ with n discretized points then the algorithm will out the DFT on $[0, \frac{(n-1)\pi}{M}]$. Therefore, in order to use FFT to obtain the price of a call with strike price k, we must choose $n = \frac{Mk}{\pi} + 1$. Now choose ϕ so that

$$\phi(\omega) = \frac{S_0^{1+\alpha+\omega i}\psi((\gamma-\omega i-\alpha-1)i)}{\psi((\gamma-1)i)(\alpha^2+\alpha-\omega^2+(2\alpha+1)\omega i)}.$$

Taking the Fourier Transform of ϕ will give $\hat{C}$, where $\hat{C}(k) = e^{\alpha k}C(e^k)$. So the call price $C(k)$ at strike price k is given by $k^{-\alpha}\hat{C}(\ln k)$. FFT will allow us to compute $\hat{C}$ efficiently.

6.2 Numerical Experiment with Levy Processes

This section considers the CRRA prices for three different Levy processes, using the algorithm described in the previous section:

1. A CGMY process with $C = 0.07, G = 10, M = .1, Y = 1.1, m = 0.2$.

2. A Normal distribution with mean 1 and variance 0.1 In this case, the CRRA price is the Black-Scholes price.

3. A Variance Gamma distribution with location parameter 0.2, stability parameter 5, asymmetry parameter 3, and scaling parameter 0.2.

The result can be seen in figure (6.1). Theoretically, the computational time of the algorithm should be of the same order as fast fourier transform: Assuming one uses the algorithm to find the price of a call at n different strike prices, where n is a power of 2, then the computation time should be $O(n \log_2 n)$. The script for the algorithm this can be found in the Appendix section (8.3).

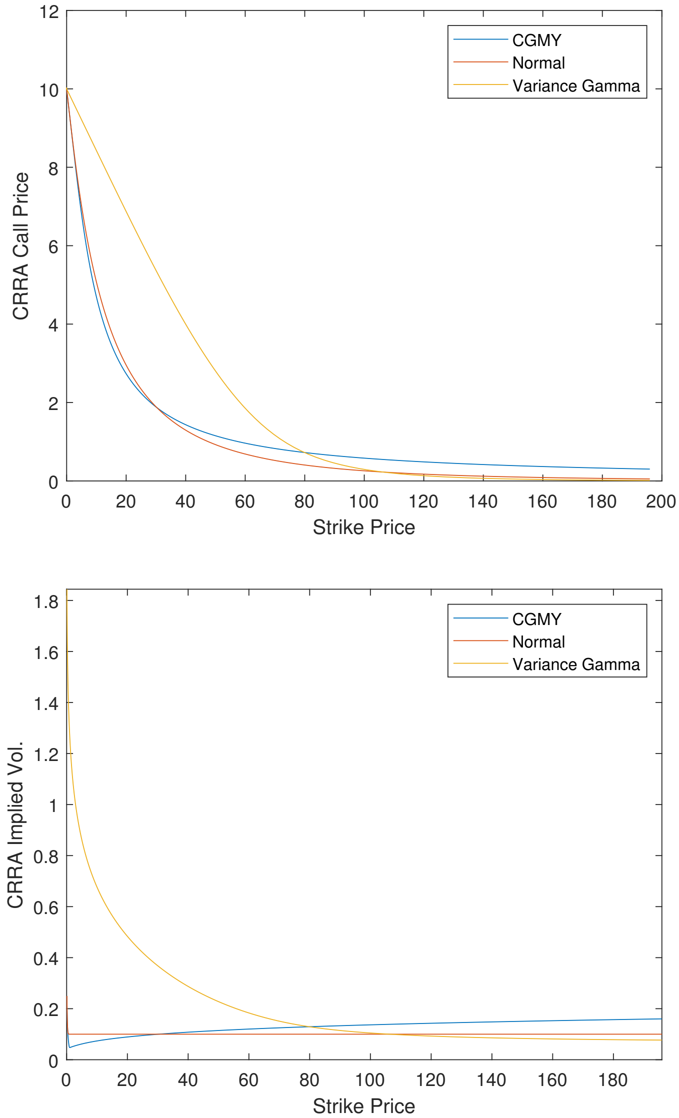

Figure 6.1: Left: The CRRA prices on a stock with underlying price 10, time to expiry 10, and interest rate 4%. Right: The corresponding implied volatilities for each process over various strike price. The CGMY and Normal implied volatilities shoot up near $K = 0$ due to numerical error from the numerical implied volatility algorithm. Observe that the implied volatility for Normal is constant.

Chapter 7

Concluding Remarks and Further Research

In summary, this thesis derives efficient methods for computing pricing bounds, as well as particular prices, on vanilla calls. This paper also derives bounds on call prices when the underlying asset is driven by a stochastic volatility process.

The original research done in this thesis can be greatly expanded. The original contributions in this thesis with regard to option bounds on stochastic volatility models can likely be expanded to more general stochastic volatility models than those discussed in this thesis. Moreover, risk-neutral processes corresponding to bounds on general Levy processes (which are more general than Jump-Diffusion processes) may be found. Computationally efficient methods can be found for models of utility beyond CRRA utility, including quadratic utility and exponential utility.

Chapter 8

Appendix

8.1 Ritchken Bounds on Discrete and Subordinated Time with Multiple Revisions

Script for Method of Moments estimator was borrowed from Rajan et al.[27]. The following is my MATLAB script for quantile estimator and bounding procedure:

```
C=.5;G=2;M=3.5;Y=.5;T=100;mu=.1;
psi=@(u,C,G,M,Y,mu)gamma(-Y).*C.*((M-1i.*u).^Y-M.^Y+1i*Y*M^(Y-1).*u)+...
gamma(-Y)*C*((G+1i.*u).^Y-G.^Y-1i.*Y.*G.^(Y-1).*u);
phi=@(t,u,C,G,M,Y,mu)exp(t.*psi(u,C,G,M,Y)).*exp(1i*mu.*u);
[cgmy_pdf,uu]=pdffromcf(@(u)phi(1,u,C,G,M,Y,mu));
cgmy_cdf=cumsum(cgmy_pdf(1:end-1).*diff(uu));
uuind=zeros(n,1);
for j=1:n
uind=find(cgmy_cdf>j/n);
uuind(j)=uind(1);
end
uu=[uu(uuind);1/n*ones(1,n)];
p=1/n*ones(1,n);
u=quantile(sp,[cumsum(p(1:end-1)) 1]);
u=[u; p];
x=2600;
s0=2883.2;
```

```
lo=zeros(10,1);up=zeros(10,1);rn=zeros(10,1);
llo=zeros(10,1);uup=zeros(10,1);rrn=zeros(10,1);bs=zeros(10,1);
r=1.03333;
bsup=(r-exp(-sqrt(uu(1,:).^2*uu(2,:)'-uu(1,:)*uu(2,:)')))....
/(exp(sqrt(uu(1,:).^2*uu(2,:)'-uu(1,:)*uu(2,:)'))...
-exp(-sqrt(uu(1,:).^2*uu(2,:)'-uu(1,:)*uu(2,:)')));
bsdp=1-bsup;
bsu=exp(sqrt(uu(1,:).^2*uu(2,:)'-uu(1,:)*uu(2,:)'));bsd=1./bsu;
lll=10;
for j=1:lll
[llo(j),uup(j)]=mppbound(s0,j,x,r,exp(uu(1,:)),uu(2,:));
bs(j)=binPriceCRR(x,s0,r,...
sqrt(uu(1,:).^2*uu(2,:)'-(uu(1,:)*uu(2,:)').^2),1/j,j,'CALL',0);
end
% 0 revision opportunities
llo=[s0-x*(s0-x>0); llo];
uup=[s0-x*(s0-x>0); uup];
%Poisson Subordinator
lambda=[1:.1:7];mm=length(lambda);
upsub=zeros(mm,1);losub=zeros(mm,1);
for j=1:mm
sub=makedist('Poisson',lambda(j));
prs=pdf(sub,[0:lll]);
upsub(j)=prs*uup;
losub(j)=prs*llo;
end
figure(6)
plot(0:lll,llo)
hold on
plot(0:lll,uup)
% hold on
% plot(1:10,bs)
xlabel('revision opportunities')
ylabel('current price')
legend('lower','upper','blackscholes')
hold off
figure(7)
plot(lambda,losub,lambda,upsub);
legend('lower','upper')
xlabel('\lambda')
```

```
ylabel('Price')
function [loback,upback,rnprice]=mppbound(s0,k,x,r,u,p)
r=r^(1/k);
u=u.^(1/k);
if nargin == 5
if k==1


[loback,upback]=opobound(s0,x,r,u);
else
h=find(u<r,1,'last');
alpha = (r-u(h))/(u(h+1)-u(h));theta=(r-u(1))/(u(end)-u(1));
cu=factorial(k)./(factorial(k-[0:k]).*factorial([0:k]))...
.*theta.^[0:k].*(1-theta).^(k-[0:k]).*...
max(0,u(end).^[0:k].*u(1).^(k-[0:k]).*s0-x);
cl=factorial(k)./(factorial(k-[0:k]).*factorial([0:k]))...
.*alpha.^[0:k].*(1-alpha).^(k-[0:k]).*...
max(0,u(h+1).^[0:k].*u(h).^(k-[0:k])*s0-x);
upback = r^(-k)*sum(cu,2)';
loback = r^(-k)*sum(cl,2)';
end
else
if k==1
[loback,upback]=opobound(s0,x,r,u,p);
else
n=k;
u=u(p~=0);
p=p(p~=0);
[u,I]=sort(u);
p=p(I);
m=length(p);
uhat=cumsum(u.*p)./cumsum(p);
uhatstring=nmultichoosek(uhat,n);
ustring=nmultichoosek(u,n);
uu=prod(ustring,2);
h=find(uhat<r,1,'last');
alhat=(r-uhat(h))/(uhat(h+1)-uhat(h));
thhat = (r-uhat(1))/(uhat(end)-uhat(1));
cub=zeros(nchoosek(m+n-2,n-1),length(x));
clb=zeros(nchoosek(m+n-2,n-1),length(x));
```

```
crn=zeros(nchoosek(m+n-2,n-1),length(x));
cf=max(0,uu.*s0-x');
uback=nmultichoosek(uhat,n-1);
%back propagate
for k=1:nchoosek(m+n-2,n-1)
ind=indlat(uback(k,:),uhat',uhatstring);
cfin=sort(cf(ind,:),1);
chat=cumsum(cfin.*p',1)./cumsum(p');
cub(k,:)=r^(-1).*(thhat.*chat(end,:)+(1-thhat).*chat(1,:));
clb(k,:)=r^(-1).*(alhat.*chat(h+1,:)+(1-alhat).*chat(h,:));
crn(k,:)=r^(-1)*chat(end,:);
end
for j=n-1:-1:2
cl=zeros(nchoosek(m+j-2,j-1),length(x));
cu=zeros(nchoosek(m+j-2,j-1),length(x));
cr=zeros(nchoosek(m+j-2,j-1),length(x));
j
ub=nmultichoosek(uhat,j-1);
for i=1:nchoosek(m+j-2,j-1)
ind=indlat(ub(i,:),uhat',uback);
cubb=sort(cub(ind,:),1);
clbb=sort(clb(ind,:),1);
crnn=sort(crn(ind,:),1);
cuhat=cumsum(cubb.*p',1)./cumsum(p');
clhat=cumsum(clbb.*p',1)./cumsum(p');
cu(i,:)=r^(-1).*(thhat.*cuhat(end,:)+(1-thhat).*cuhat(1,:));
cl(i,:)=r^(-1).*(alhat.*clhat(h+1,:)+(1-alhat).*clhat(h,:));
cr(i,:)=r^(-1)*p*crnn;
end
clb=cl;
cub=cu;
crn=cr;
uback=ub;
end
cuhat=cumsum(cub.*p',1)./cumsum(p');
clhat=cumsum(clb.*p',1)./cumsum(p');
upback=r^(-1).*(thhat.*cuhat(end,:)+(1-thhat).*cuhat(1,:));
loback=r^(-1).*(alhat.*clhat(h+1,:)+(1-alhat).*clhat(h,:));
rnprice=r^(-1)*p*crn;
end
```

```
end
end

function combs = nmultichoosek(values, k)
%// Return number of multisubsets or actual multisubsets.
if numel(values)==1
n = values;
combs = nchoosek(n+k-1,k);
else
n = numel(values);
combs = bsxfun(@minus, nchoosek(1:n+k-1,k), 0:k-1);
combs = reshape(values(combs),[],k);
end
end

function h=indlat(uback,u,ustring)
uback=[repmat(uback,length(u),1) u];
h=ismember(sort(ustring,2),sort(uback,2),'rows');
end

function [f,u]=pdffromcf(c)
[f,u]=fastft(c,10,1000);
end

function [fhat,u]=fastft(f,M,N)
dx=2*M/N;
x=[-M:dx:M];
fd=f(x);
fhat=M/(pi*N)* (fftshift(fft(ifftshift(fd))));
du=pi/M;
u=-N/2*pi/M+[0:N-1]*du;
fhat=fhat(abs(u)<N);
u=u(abs(u)<N);
end
```

8.2 Ritchken Bounds with FFT

```
K=200; s=100; r=.03; T=1/12; % set strike
```

```
C=.02;G=.03;Y=1.7;M=.02;
mu=(log(s)+.1);sigma=.1;
psi1=@(u)exp(mu.*u.*1i-sigma.^2./2.*u.^2); % test cf
psi2=@(u)cgmy_cf(u,C,G,M,Y).*exp(1i.*u.*mu);
[yu,yl,ku,kl]=rbfft(psi2,K,s,r,T);
merton=s-exp(-r*T)*kl;
merton=merton.*(merton>0);
figure(1)
plot(ku,yu)
hold on
plot(kl,yl)
plot(kl,merton)
legend('upper bound','lower bound','Merton bound')
title('CGMY Bounds')
xlabel('Strike Price')
ylabel('Call Price')
hold off
[yu,yl,ku,kl]=rbfft(psi1,K,s,r,T);
merton=s-exp(-r*T)*kl;
merton=merton.*(merton>0);
figure(2)
plot(ku,yu)
hold on
plot(kl,yl)
hold on
plot(kl,merton)
xlabel('Strike Price')
ylabel('Call Price')
legend('upper bound','lower bound','Merton bound')
title('Gaussian Bounds')
hold off

function [yu,yl,uu,ul]=rbfft(psi,K,s,r,T)
%% lower bound
r=exp(r*T);
N=5000;
M=500;
alpha=1e-5i;
x=log(s);
k=log(K);
```

```
phi=@(u)psi(u)./(1i.*u);
[z1,x1]=fastft(@(u)phi(u+alpha.*1i),M,N);
z1=(exp(alpha.*x1).*z1)*2*pi;
phi=@(u)psi(u)./(1-1i.*u);
[z2,x2]=fastft(@(u)phi(u),M,N);
z2=2*pi*z2.*exp(x2);
d=r*pi-r*real(z1);
g=(d*s-real(z2));
xind=find(abs(g)==min(abs(g(x1>x+log(r)))));
xind=xind(end);
xs=x1(xind);
c2=real(z2(xind));
c1=real(z1(xind));
kind=find((x1<=k));
p1=real(z1(kind));
p2=real(z2(kind));
K=exp(x1(kind));
yl=(c2+(K).*c1-(K).*p1-p2)./d(xind).*(K<exp(xs));
ul=K;
%% upper bound
N=10000;
M=100;
alpha=3e-2;
phi=@(u)psi(u-(alpha+1)*1i)./(alpha.^2+alpha-u.^2+1i.*(2.*alpha+1).*u);
[yu,xu]=fastft(phi,M,N);
yu=exp(-alpha.*xu).*yu;
kind=(xu<=k);
Ku=exp(xu(kind));
yu=real(yu(kind));
yu=s*yu./psi(-1i);
uu=Ku;
end
```

8.3 CRRA Price with FFT

```
[y1,k1]=fastprice(@(u)exp(2i.*u).*cgmy_cf(u,.5,3,5,1.7), 10,20,.04,10,100);
[y2,k2]=fastprice(@(x)exp(x.*.1i-x.^2./20),10,20,.04, 10,100);
[y3,k3]=fastprice(@(u)variance_gamma_cf(u,.5,.1,2,2), 10,20,.04, 10,100);
plot(k1,y1)
```

```
hold on
plot(k2,y2)
hold on
plot(k3,y3)
hold off
xlabel('Strike Price')
ylabel('CRRA Call Price')
legend('CGMY','Normal','Variance Gamma')
shg

function [C,k]=fastprice(psi,s,k,r,T,n)
[gam,err] = lsqnonlin(@(gam)psi((real(gam-1 ).*1i))-exp(r*T).*psi(real(gam).*1i),1);
gam=real(gam);
pii=@(alp,u)1./(psi((gam-1)*1i))...
.*s.^(1+alp+u.*1i).*psi(( gam-u*1i-alp-1).*1i)...
./( alp.^2+alp-u.^2+(2*alp +1).*u.*1i);
alpspan=[.1:.1:10];m=length(alpspan);
cfs=zeros(m,201);
alps=[];
uspan=[-10:.1:10];
for j=1:m
cfs(j,:)=pii(alpspan(j),uspan);
if ~isempty(intersect(real(cfs(j,abs(uspan)<9)),max(real(cfs(j,:)))))...
&& sum(isnan((cfs(j,:))))+sum((isinf(cfs(j,:))))==0
alps=[alps alpspan(j)];
end
end
alp=.1;
[C,k]=roseprice(psi,s,k,n,alp,gam);
end

function [y,K]=roseprice(psi,s,k,n,alp,gam)
kk=log(k);
pi=@(alp,u)1./(psi((gam-1)*1i))...
.*s.^(1+alp+u.*1i).*psi(( gam-u*1i-alp-1).*1i)...
./( alp.^2+alp-u.^2+(2*alp +1).*u.*1i);
phi=@(u)pi(alp,u);
[y,uu]=(fastft(phi,n,10000));
y=y(uu<kk);
uu=uu(uu<kk);
```

```
y=exp(−uu.∗alp).∗abs(y);
K=exp(uu);
end
```

8.4 Stochastic Volatility Bounds

call_heston_cf, which is for pricing options under the Heston model, was borrowed from Cristomos[10]. The following script is for stochastic dominance bounds using call_heston_cf:

```
K=(0:200);lb=zeros(length(K),1);ub=zeros(length(K),1);heston=zeros(length(K),1);
s=100;v=.1;t=1;vbar=.2;a=.3;eta=1;rho=−0.3;r=.04;mu=.1;
for k=1:length(K)
[ub(k),lb(k),heston(k)]=heston_dara_bounds(s,v,t,vbar,a,eta,rho,r,mu,K(k),−.2);
end
uvol=calcBSImpVol(1,ub,s,K',t∗ones(length(K),1),r,0);
lvol=calcBSImpVol(1,lb,s,K',t∗ones(length(K),1),r,0);
hestvol=calcBSImpVol(1,heston,s,K',t∗ones(length(K),1),r,0);
iv=(s−exp(−r∗t).∗K).∗(s−exp(−r∗t).∗K>0);
figure(1)
plot(K,lb,'b',K,ub,'r',K,iv,K,heston,'k−−')
legend('lower bound','upper bound','Merton bound','Heston with \lambda=−0.2')
xlabel('strike price')
ylabel('call price')
hold off
figure(2)
plot(K,lvol,'b',K,uvol,'r',K,hestvol,'k−−')
legend('lower bound','upper bound','Heston with \lambda=−0.2')
xlabel('strike price')
ylabel('implied volatility')

function [ub,lb,heston]=heston_dara_bounds(s,v,t,theta,kappa,sigma,rho,r,mu,K,lambda)
% s = current price
% v = current volatility
% t = time to maturity
% theta = long−term mean volatility
% kappa = volatility mean reversion rate
% sigma.^2 = volatlity of volatility
% rho = price−volatility correlation
% mu = drift parameter
```

```
% K = strike price
% lambda = risk-aversion parameter
ub=call_heston_cf(s, v, theta-(mu-r)*sigma/kappa/rho, kappa, sigma, r, rho, t, K);
lb=call_heston_cf(s, v, theta, kappa, sigma, r, rho, t, K);
heston = call_heston_cf(s, v, theta*kappa/(kappa+lambda), kappa+lambda, sigma, r, rho, t, K);
end
```

8.5 Data

S&P500 stock data was collected on October 5, 2018 from Yahoo!Finance. Daily adjusted close price was used for parameter estimation. Ford stock data was collected on October 15, 2018 from Yahoo!Finance. Monthly adjusted close price was used for non-volatility stock price parameter estimation. Daily adjusted close price was used for volatility estimation and volatility-related parameter estimation.

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