DERIVATION OF NEWTON'S SECOND LAW FOR A CHARGED PARTICLE BASED ON COULOMB'S LAW AND FINITE ELECTROSTATIC PROPAGATION SPEED

ABSTRACT

This paper mathematically derives the basic form of Newton's second law, that inertial force *F* is proportional to mass *m* times acceleration *a*, for a charged particle (proton or electron), based on four assumptions: 1) the particle is theoretically modeled as a hollow spherical shell of charged matter; 2) the shell's radius is defined by "classical electron radius" theory; 3) every segment of the shell is repelled from every other segment of the shell by an electrostatic force defined by Coulomb's Law; and 4) the electrostatic field propagates at a finite speed. The inertial force is due to the fact that any segment *i* of the charged particle will perceive any other segment *j* of the particle that is ahead of *i* in the direction of acceleration, to be closer than segment *j* perceives segment *i*. This leads to an imbalance of Coulombic forces imparted to different segments of the particle, yielding a resultant force $\mathbf{F} = [\frac{2}{3}c^2/v^2]ma$, *where* c/v is a constant equaling speed of light over speed of the electrostatic propagation. Although the derivation takes into account the particle's radius *r* and charge *q*, those parameters cancel out.

DERIVATION

Stationary Charged Shell

Consider a charged particle – proton or electron – modeled as a hollow spherical shell of charged matter, as it is in "classical electron theory".[1,2] For analysis purposes, consider the sphere centered on the origin of a θϕr globe-type polar coordinate system like that of Figure 1. The origin is on the polar axis $\bar{x}$, *r* is the shell's radius, θ is a longitudinal angle with θ=0 and π located on $\bar{x}$ at the shell's two poles designated A and B, and ϕ is a latitudinal angle about the polar axis. This polar coordinate system is related to an xyz Cartesian coordinate system by: $x = r\cos\theta$, $y = r\sin\theta\sin\phi$, and $z = r\sin\theta\cos\phi$. The shell is sectioned with a geological grid: $2n$ longitude lines divide the shell into $2n$ slices, each having a latitudinally-extending arcuate width of $\pi r\sin\theta/n$. Latitude lines divide the shell into *n* rings surrounding $\bar{x}$, with each ring having an arcuate width of $\pi r/n$. The $2n$ slices and *n* rings define $2n^2$ shell segments, each segment *i* having a surface area $\pi^2r^2\sin\theta_i/n^2$.

If charge density is uniform over the entire shell, then the ratio of charge q_i of each segment *i* to the shell's total charge q equals the ratio of the surface area $\pi^2r^2\sin\theta_i/n^2$ of the segment *i* to the shell's total surface area $4\pi r^2$. Therefore:

$$q_i = q\pi\sin\theta_i/4n^2 \qquad (1)$$

Each segment i of the shell is electrostatically repelled from every other segment j of the shell by the Coulombic force:

$\mathbf{F}_{ij} = kq_iq_j/d_{ij}^2$ (2)

where $\mathbf{F}_{ij}$ is directed from segment j toward segment i, and d_{ij} is the distance from segment i to segment j, and

$d_{ij} = x_{ij}^2+y_{ij}^2+z_{ij}^2$, (3)

$x_{ij} = r\cos\theta_j - r\cos\theta_i$ (4)

$y_{ij} = r\sin\theta_j \sin\phi_j - r\sin\theta_i \sin\phi_i$ (5)

$z_{ij} = r\sin\theta_j \cos\phi_j - r\sin\theta_i \cos\phi_i$ (6)

Since segment i is repelled from segment j by the same force $\mathbf{F}_{ij}$ that j is repelled from i, for each pair of the particle's segments, the forces are balanced, so the particle exerts no net force.

Accelerating the Charged Shell

Now consider the shell being forced to accelerate at a constant acceleration rate a along $\vec{x}$, as indicated by the arrow in Figure 1, while maintaining its spherical shape and original radius r. The value of acceleration a is the same no matter what inertial reference frame it is measured against (in contrast to speed, whose value depends on the inertial frame it is measured against), including relative to the inertial frame of the particle itself. However, the particle's speed remains zero relative to its own current inertial frame of references.

How can the particle be constantly accelerating relative to its own inertial reference frame if its speed remains zero relative to that inertial frame? The answer is facilitated by considering the motion of a ball bouncing off the ground directly upward and reaching an apex before falling directly back to the ground. Throughout the entire trajectory, while both rising and falling, the ball is successively assuming an infinite series of consecutive inertial frames while it retains a constant downwardly-directed acceleration a of 9.8 m/s. At each moment, it is at rest relative to its "current" inertial frame, and remains in that current inertial frame for only an infinitesimal time period dt before progressing to the next inertial frame. At each moment, its speed is zero relative to its "current" inertial frame, but is $a{\cdot}dt$ relative to its previous inertial frame, and is $2{\cdot}a{\cdot}dt$ relative to its inertial frame before that. Consequently, even though the ball's speed is constantly changing from the perspective of the ground, the ball appears from its own perspective to be in a steady state in which it has both constant (specifically zero) speed and constant acceleration relative to its own current inertial frame at each moment.

Acceleration's Effect on the Segments' Repulsive Forces

Now consider a pair of segments *i* and *j* of the accelerating shell, as shown in Figure 2. Both segments accelerate in the $\bar{x}$ direction, with segment *j* farther ahead than segment *i*. Assume that electrostatic charge propagates at a finite speed *v*, as proposed by Turtur[3,4]. From segment *i*'s perspective, the electrostatic field from every segment, including segment *j*, approaches it at speed *v*. So even though segment *j* is, at time t=0 (and in fact at all times), at a point in space a distance d_{ij} from segment *i*, that point in space electrostatically appears to segment *i* at t=0 to be unoccupied, because segment *j* was not there at time $t=-d/v$. To segment *i* at t=0, segment *j* instead electrostatically appears to be where it was at time $t=-d/v$, which is at a differential distance $\Delta_x = \frac{1}{2}at^2 = \frac{1}{2}ad_{ij}^2/v^2$ to the right of segment *j*'s position at t=0. Since Δ_x is in the $\bar{x}$ direction, the shell's acceleration causes segment *i* to perceive its distance d_{ij} from segment *j* to be **shortened** in the $\vec{d}_{ij}$ direction (assuming $\Delta_x \ll d_i$) by a distance

$$\Delta_{ij} = \Delta_x\,(x_{ij}/d_{ij}) = \tfrac{1}{2}ax_{ij}\,d_{ij}/v^2 \qquad (7)$$

Since acceleration *a* is constant in this scenario, segment *i* perceives segment *j* to be at a fixed location a distance d_{ij} **-** Δ_{ij} away. This concept that electrostatic field, emitted by one segment of an electron's hypothetical charged spherical shell, takes a finite transit time to reach another segment of the shell is similar to that described by Jackson[5] with respect to the finite transit time it takes light radiated by one segment of the electron shell to reach the opposite end of the electron shell. The concept that this finite transit time would lead to aberration of the apparent position of an electrostatic segment corresponds to the well-known relativistic aberration of a moving light source's apparent position due to light's finite transit time[6].

Similarly, segment *j* at t=0 perceives the distance d_{ij} to segment *i* to be **lengthened** by the same distance Δ_{ij}, and continuously perceives segment *i* to be at a fixed location a distance $d_{ij} + \Delta_{ij}$ away (per Figure 2).

Since, electrostatically, segment *i* perceives its distance from *j* to be shorter than segment *j* perceives its distance from *i*, the particle experiences is an imbalance of repulsive electrostatic forces. Although this imbalance in forces occurs in a single snapshot in time t=0, its magnitude would be the same no matter which time *t* the snapshot is taken, since each segment perceives the other segments to be motionless (in accordance with the steady state scenario explained above) but in a different position than if the particle was not accelerating.

The resultant force of the two segments *i* and *j*, simultaneously repelled from each other at the same time t=0, equals the repulsive force urging *i* away from *j* minus the repulsive force urging *j* away from *i*. This converts Eq. 2 to:

$$\mathbf{F_{ij}} = \frac{kq_i q_j}{(d_{ij} - \Delta_{ij})^2} - \frac{kq_i q_j}{(d_{ij} + \Delta_{ij})^2} \qquad (8)$$

Assuming $d_{ij} >> \Delta_{ij}$ yields:

$\mathbf{F}_{ij} = kq_i q_j\,[4\Delta_{ij} d_{ij}^{-3}]$

Substituting $q_i = q\pi\sin\theta_i/4n^2$ and $q_j = q\pi\sin\theta_j/4n^2$ from Eq. 1 and $\Delta_{ij} = \frac{1}{2}ax_{ij}\,d_{ij}/v^2$ from Eq. 7 yields:

$\mathbf{F}_{ij} = a k q^2 \pi^2 \sin\theta_i \sin\theta_j\, x_{ij}/(8v^2 d_{ij}^2 n^4)$ (9)

Since segments *i* and *j* perceive each other to be fixed in space, in accordance with the steady state scenario explained above but merely at different locations than if the particle were not accelerating, $\mathbf{F_{ij}}$ does not vary with time, and no magnetic flux is induced in accordance with Maxwell's equations that would affect $\mathbf{F_{ij}}$.

The resultant force $\mathbf{F}_{ij}$ of the segment pair is aligned in the $\vec{d}_{ij}$ direction, from *j* toward *i*. Since the shell and its acceleration $\vec{a}$ are symmetric about $\vec{x}$, the $\vec{y}$ and $\vec{z}$ components of $\mathbf{F}_{ij}$ cancel out when summed over all segment pairs of the shell, so the $\vec{x}$ component of $\mathbf{F}_{ij}$ is the only component of consequence. The $\vec{x}$ component of $\mathbf{F}_{ij}$ equals

$\mathbf{F}_{ij\mathrm{x}} = \mathbf{F}_{ij}\,(x_{ij}/d_{ij})$.

Substituting for $\mathbf{F}_{ij}$ from Eq. 9 yields:

$\mathbf{F}_{ij\mathrm{x}} = a k q^2 \pi^2 \sin\theta_i \sin\theta_j\, x_{ij}^2/(8v^2 d_{ij}^3 n^4)$ (10)

Summing $\mathbf{F}_{ij\mathrm{x}}$ over all of the shell's pairs of segments yields:

$$\mathbf{F} = \left[\frac{akq^2}{v^2 r}\right] \sum_{i}^{2n^2} \sum_{j=i}^{2n^2} \left[\frac{\pi^2 \sin\theta_i \sin\theta_j (x_{ij}/r)^2}{8n^4 (d_{ij}/r)^3}\right] \qquad (11)$$

where, per Eqs. 3-6:

$x_{ij}/r = \cos\theta_j - \cos\theta_i$ and

$d_{ij}/r = [x_{ij}^2 + (\sin\theta_j\sin\phi_j - \sin\theta_i\sin\phi_i)^2 + (\sin\theta_j\cos\phi_j - \sin\theta_j\cos\phi_j)^2]^{1/2}$

The double summation $\Sigma\Sigma[\pi^2\sin\theta_i\sin\theta_j(x_{ij}/r)^2/[8n^4(d_{ij}/r)^3]$ in Eq. 11 was determined through finite element analysis, by considering the charge q_i and q_j of each segment *i* and *j* as concentrated at the segment's center point per Figure 1. The error resulting from this assumption approaches zero as segment size approaches zero as *n* approaches ∞. The double summation for $n = 40$, 80 and 160 was calculated to be 0.325294, 0.329271, and 0.331292, respectively. Plotting these values against $1/n$ yields a straight line with a y-intercept, at $1/n = 1/\infty$, of $\frac{1}{3} \pm .00002$. Substituting $\frac{1}{3}$ for the double summation in Eq. 11 yields:

$F = akq^2/(3v^2 r)$ (12)

Determining the Shell's Radius

The shell's radius r in Eq. 12 is determined in accordance with "classical electron radius" theory[7], wherein the shell's rest energy mc^2 equals the shell's electrostatic potential energy:

$$E_{\text{potential}} = \sum_{i}^{2n^2} q_i V_i = q\sum_{i}^{2n^2} V_i \qquad (13)$$

where q_i is the charge of the i^{th} segment of the particle of charge q. In Eq. 13, the electrical potential V_i at any segment i on the shell is the same as at pole A, which equals:

$$V_{i=A} = \sum_{j}^{2n^2} \frac{kq_j x_j}{d_j^2} = \sum_{j}^{2n^2} \frac{kq_j \pi(\sin\theta_j)x_j}{4n^2 r(x_j^2 + y_j^2 + z_j^2)}$$

Substituting this into Eq. 13 yields:

$$E_{\text{potential}} = \left[\frac{kq^2}{r}\right]\sum_{j}^{2n^2} \frac{\pi(\sin\theta)x_j}{4n^2(x_j^2 + y_j^2 + z_j^2)} \qquad (14)$$

The summation $\Sigma[\pi x_j \sin\theta_j/(4n^2(x_j^2+y_j^2+z_j^2))]$ in Eq. 14 was calculated for $n = 40$, 80 and 160 to be 0.500128, 0.500032 and 0.500008, respectively. Plotting these summation values against $1/n^2$ yields a straight line with a y-intercept, at $1/n^2 = 1/\infty$, of $\frac{1}{2}$. Substituting $\frac{1}{2}$ for the summation in Eq. 14 yields:

$E_{\text{potential}} = kq^2/(2r)$, which is in agreement with Jackson[8].

Equating this to the shell's rest energy mc^2 yields

$mc^2 = kq^2/(2r) \Rightarrow r = kq^2/(2mc^2)$

Substituting this value of r into Eq. 12 yields:

$$\mathbf{F} = [\tfrac{2}{3}c^2/v^2]ma \qquad (15)$$

Discussion

The resulting Eq. 15 states that inertial force **F**, resisting an acceleration a of a charged particle of mass m relative to its own frame of reference, is derived above to be $[\frac{2}{3}c^2/v^2]ma$, where c and v are the speeds of light and electrostatic field, respectively. This agrees with the basic form of Newton's second law, that force is proportional to mass times acceleration. Eq. 15 simplifies to $F=ma$ if $v^2 = \frac{2}{3}/c^2$ *i.e.*, if $v \sim .82c$. The fact that v might be equal to or less than c is proposed by Turtur[3,4].

This derivation is based on theoretically modeling the charged particle as a hollow spherical shell of charged material with a radius defined by "classical electron radius" theory[1,2,7], in which every segment of the shell electrostatically repels every other segment of the shell. This hollow shell model was chosen for this derivation only because it simplifies the integrations of Eqs. 11 and 13. Basing the derivation on a differently shaped model, such as a cloud of charged material, would probably yield a coefficient different than $\frac{2}{3}c^2/v^2$. Although this derivation applies only to charged particles – protons and electrons – it might also apply to neutrons if a neutron consists of both a proton and an electron as proposed by Chadwick[9].

This derivation bases inertial force on an imbalance of internal electrostatic repulsive forces applied by the accelerating particle's own segments on each other due to finite speed of electrostatic propogation. This is in sharp contrast to another reported derivation[10] of inertial force that is based on external gravitational attractive forces exerted by all matter of the universe on the accelerating mass.

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[1] Wikipedia article entitled "Classical electron radius" at http://en.wikipedia.org/wiki/Classical_electron_radius

[2] John David Jackson, *Classical Electrodynamics* (John Wiley & Sons, New York (1975), p.797

[3] C. W. Turtur, *Conversion of the Vacuum-energy of Electromagnetic Zero Point Oscillations into Classical Mechanical Energy*, The General Science Journal (May, 2009), www.wbabin.net/physics/turtur1e.pdf, p. 93.

[4] C. W. Turtur, *About the Electrostatic Field following Coulomb's law with Additional Consideration of the Finite Speed of Propagation Following the Theory of Relativity* (Nov. 2009), http://philica.com/display_article.php?article_id=112

[5] John David Jackson, *Classical Electrodynamics* , John Wiley & Sons, New York (1975), p.797

[6] Wikipedia article entitled "Aberration of light" at http://en.wikipedia.org/wiki/Aberration_of_light

[7] ibid., p. 681.

[8] ibid, p. 46; equation 1.51.

[9] J. Chadwick, *The Existence of a Neutron*, Proc. Roy. Soc., A, 136 (1932), p. 692-708, available at http://www.chemteam.info/Chem-History/Chadwick-1932/Chadwick-neutron.html.

[10] A. K. T. Assis, *Relational Mechanics*,Apeiron, Montreal, Canada (1999), http://www.ifi.unicamp.br/~assis/, pp. 173-178.

FIGURE 1
SPHERICAL SHELL CONFIGURATION

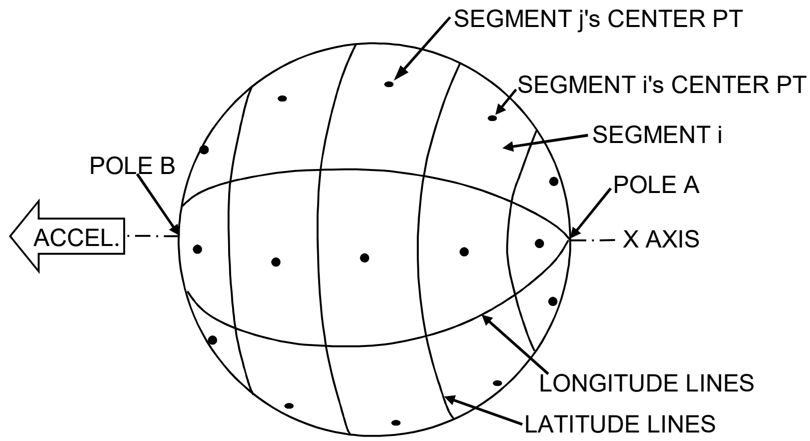

FIGURE 2
RELATIVE POSITIONS OF SEGMENTS *i* AND *j*